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On the use of some tangent cones and sets in vector optimization

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Abstract

For various classes of vector optimization problems, necessary and sufficient optimality conditions are developed in terms of first order tangent cones and second order tangent sets and cones. Additional remarks are also made.

Key words

Vector optimization problems, efficient points, optimality conditions,
tangent cones, tangent sets.

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1 Introduction, notations and preliminaries

First order tangent cones and second order tangent cones and sets play a central role, not only in scalar optimization problems but also in vector optimization problems. In the present paper we point out some uses and applications of some first and second order tangent cones and of second order tangent sets, in obtaining necessary and sufficient optimality conditions for various types of vector optimization problems, under different differentiability assumptions. The paper is organized as follows. Section 1 is the Introduction; Section 2 is concerned with the use of some first and second order tangent sets to obtain optimality conditions for a vector optimization problem with a set constraint. Section 3 is concerned with optimality conditions, via tangent cones and sets, under twice differentiability, whereas Section 4 develops optimality conditions under Hadamard differentiability. Section 5 is concerned with a multiplier rule for a nonsmooth problem. The last Section 6 makes some comments on a "gap" between scalar and vector optimization, with reference to the Guignard-Gould-Tolle constraint qualification.

We consider a nonempty set E of a partially ordered real space Y (e. g. $Y = \mathbb{R}^n$).

Definition 1.

A binary relation $\leq$ on Y is called a *partial ordering* on Y if the following properties are satisfied (for arbitrary $x,\ y,\ z,\ u \in Y$ and $\alpha \in \mathbb{R}_+$):

(i) $x \leq x$;

(ii) $x \leq y,\ y \leq z \Longrightarrow x \leq z$;

(iii) $x \leq y,\ u \leq z \Longrightarrow x + u \leq y + z$;

(iv) $x \leq y \Longrightarrow \alpha x \leq \alpha y$.

A partial ordering is called *antisymmetric* if the following condition holds: $x \leq y,\ y \leq x \Longrightarrow x = y$.

Definition 2.

A real linear space equipped with a partial ordering is called a *partially ordered linear space.*

Here we give a characterization of a partial ordering in a real linear space.

Theorem 1.

(i) If $\leq$ is a partial ordering on Y, then the set

$$D = \{x \in Y : 0 \leq x\}$$

is a convex cone. If, in addition, $\leq$ is antisymmetric, then D is pointed.

(ii) If D is a convex cone in Y, then the binary relation

$$x \leq_D y \Longleftrightarrow y - x \in D$$

is a partial ordering on Y. If, in addition, D is pointed, then $\leq$ is antisymmetric.

We recall that a nonempty set $D \subset Y$ is a *cone* if $x \in D, \alpha \geq 0 \Longrightarrow \alpha x \in D$ (some authors require only $\alpha > 0$). A cone D is *pointed* if $D \cap (-D) = \{0\}$.

The most usual ordering cone in a finite dimensional space $\mathbb{R}^n$ is given by the non negative orthant $\mathbb{R}^n_+$. This set is a pointed, closed and convex cone that defines the componentwise partial ordering on $\mathbb{R}^n$, also called a *Pareto* order. We shall be mainly concerned with this case.

One of the most important applications of the vectorial optimization techniques is found in the study of vectorial mathematical programming problems, and, as a particular case, in multiobjective programming problems. In general, if X, Y, Z and W are partially ordered linear spaces, with $D \subset Y$ the ordering cone on Y, $K \subset Z$ the ordering cone on Z and $M \subset X$ a nonempty set, we may consider a generic vectorial optimization problem

$$Min\ f(x), \text{ subject to } x \in M, \tag{1}$$

where $f : X \longrightarrow Y$, and a constrained vectorial optimization problem

$$Min\ f(x), \text{ subject to } x \in M, \tag{2}$$

where $M = S \cap Q$,

$$S = \{x \in X : g(x) \in -K,\ h(x) = 0\}, \quad Q \subset X,$$
$$f : X \longrightarrow Y,\ g : X \longrightarrow Z,\ h : X \longrightarrow W.$$

The set Q is also called "set constraint" or "abstract constraint". Usually $Q = X$.

When (2) is specialized to a multiobjective (nonlinear) programming problem, we have the formulation

$$Min\ f(x), \text{ subject to } x \in M, \tag{3}$$

where $M = S \cap Q$,

$$S = \{x \in \mathbb{R}^n : g(x) \leq_K 0,\ h(x) = 0\}, \quad Q \subset \mathbb{R}^n,$$
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^p,\ g : \mathbb{R}^n \longrightarrow \mathbb{R}^m,\ h : \mathbb{R}^n \longrightarrow \mathbb{R}^r.$$

If $D = \mathbb{R}^p_+$ and $K = \mathbb{R}^m_+$, we have the classical *multiobjective Pareto problem or vector Pareto problem*. Finally, if $p = 1$ and $D = \mathbb{R}_+$, (3) collapses to the usual scalar nonlinear programming problem.

Now we give the solution concepts for a vector optimization problem. We consider only efficient and weak efficient solutions (see, e. g., Ehrgott (2005), Ehrgott and Gandibleaux (2002), Luc (1989), Jahn (2005), Miettinen (1999), Sawaragi and others (1985)), but in vector optimization there are other solution concepts, such as proper efficient points (for a survey on the various definitions of proper efficient points, see the paper of Guerraggio, Molho and Zaffaroni (1994)). We consider problem (1), with $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^p, M \subset \mathbb{R}^n$ and the ordering cone $D \subset \mathbb{R}^p$ convex, closed, pointed and with $int(D) \neq \varnothing$.

Definition 3.

A point $x^0 \in M$ is said to be a *global efficient point* of (1) if, for each $x \in M$,

$$f(x^0) - f(x) \notin D \setminus \{0\}$$

i. e.

$$f(x) \notin f(x^0) - (D \setminus \{0\})$$

i.e.

$$f(x) - f(x^0) \cap (-D) = \{0\}\,.$$

A point $x^0 \in M$ is said to be a *global weak efficient point of (1), if, for each* $x \in M$,

$$f(x^0) - f(x) \notin int(D)$$

i.e.

$$f(x) \notin f(x^0) - int(D)$$

i.e.

$$f(x) - f(x^0) \cap (-int(D)) = \{0\}\,.$$

If the previous conditions are verified in $N(x^0) \cap M$, where $N(x^0)$ is a suitable neighborhood of x^0, then x^0 is said to be a *local* efficient point or a *local* weak efficient point, respectively. If $D = \mathbb{R}^p_+$ we have the *Paretian case.*

In the scalar case, i.e. with $p = 1$, we have $D \setminus \{0\} = int(D) = \mathbb{R}_+ \setminus \{0\}$, so that the previous definitions collapse to the ordinary definition of a local or global minimum point. Obviously, (local) efficiency implies (local) weak efficiency, so it is usual to give the necessary optimality conditions for weak efficient points and the sufficient optimality conditions for efficient points. From now on we consider real finite-dimensional spaces and consider the following tangent cones and sets.

Definition 4.

Let $M \in \mathbb{R}^n$ and $x^0 \in cl(M), v \in \mathbb{R}^n$.

(a) The *tangent cone* to M at x^0 (or *Bouligand tangent cone or contingent cone* to M at x^0) is given by

$$T(M, x^0) = \{y \in \mathbb{R}^n : \exists\, \{\lambda_n\} \subset \mathbb{R},\ \exists\, \{x^n\} \subset M,\ \lambda_n \longrightarrow +\infty,\ x^n \longrightarrow x^0 \\ \text{such that } \lambda_n(x^n - x^0) \longrightarrow y\}\,.$$

Equivalently:

$$T(M, x^0) = \left\{y \in \mathbb{R}^n : \exists t_n \longrightarrow 0^+,\ \exists y^n \longrightarrow y \text{ such that } x^0 + t_n y^n \in M, \forall n \in \mathbb{N}\right\}.$$

(b) The *interior tangent cone* to M at x^0 (or *cone of the interior directions* to M at x^0: see, e. g., Bazaraa and Shetty (1976)) is given by

$$TI(M,x^0) = y \in \{\mathbb{R}^n : \exists\delta > 0 \text{ such that } \\ x^0 + ty' \in M,\ \forall t \in (0,\delta),\ \forall y' \in N(y,\delta)\}\,.$$

Equivalently:

$$TI(M,x^0) = \{y \in \mathbb{R}^n : \forall t_n \longrightarrow 0^+,\ \exists y^n \longrightarrow y,\ x^0 + t_n y^n \in M, \\ \text{for } n \text{ large enough } \}\,.$$

(c) The *second order tangent set* to M at (x^0, v) is given by

$$T^2(M,x^0,v) = \{w \in \mathbb{R}^n : \exists t_n \longrightarrow 0^+,\ \exists w^n \longrightarrow w \text{ such that } \\ x^n = x^0 + t_n v + \tfrac{1}{2}t_n^2 w^n \in M,\ \forall n \in \mathbb{N}\big\}\,.$$

(d) The *asymptotic second order cone* to M at (x^0, v) is given by

$$T''(M,x^0,v) = \{w \in \mathbb{R}^n : \exists (t_n, r_n) \longrightarrow (0^+,0^+),\ \exists w^n \longrightarrow w \text{ such that } \\ (t_n/r_n) \longrightarrow 0,\quad x^n = x^0 + t_n v + \tfrac{1}{2}t_n r_n w^n \in M,\ \forall n \in \mathbb{N}\big\}\,.$$

The tangent cone T, the interior tangent cone TI and the second order tangent set T^2 (which is not necessarily a cone) are well-known. See, e. g. the paper of Giorgi, Jimenez and Novo (2010). The asymptotic second order tangent cone T'' has been independently introduced by Penot (1998) and by Cambini, Martein and Vlach (1999) in order to state optimality conditions in scalar optimization. We now collect some properties of these first and second order tangent sets and cones. First note that $T(M,x^0) = \{0\}$ if and only if x^0 is an isolated point and in such a case x^0 is obviously both a weak efficient point and an efficient point for problem (1).

Theorem 2.

Let $M \subset \mathbb{R}^n$ be a convex set and $x^0 \in cl(M)$. Then we have

(i) $T(M,x^0) = cl(cone(M - x^0))$.

If, moreover, $int(M) \neq \varnothing$, then

(ii) $TI(int(M),x^0) = TI(M,x^0) = int(cone(M - x^0)$.

(iii) $cl(TI(M,x^0)) = T(M,x^0)$.

If, moreover, M is a cone, then

(iv) $TI(M,0) = TI(int(M),0) = int(M)$.

Theorem 3.

Let M be a subset of $\mathbb{R}^n$ and let $x^0 \in cl(M), v \in \mathbb{R}^n$.

(i) $T^2(M,x^0,v)$ and $T''(M,x^0,v)$ are closed sets contained in $cl\,\{cone\,[cone(M - x^0) - v]\}$ and $T''(M,x^0,v)$ is a cone.

(ii) If $v \notin T(M,x^0)$, then $T^2(M,x^0,v) = T''(M,x^0,v) = \varnothing$.

(iii) $T^2(M,x^0,0) = T''(M,x^0,0) = T(M,x^0)$.

Theorem 4.

Let $M \subset \mathbb{R}^n$ be a convex set, $x^0 \in M, v \in T(M, x^0)$. Then:
(i) $T^2(M, x^0, v) + T(T(M, x^0), v) \subset T^2(M, x^0, v)$.
(ii) $T(T(M, x^0), v) = cl\,\{cone\,[cone(M - x^0) - v]\}$.
(iii) If $T''(M, x^0, v) \neq \varnothing$, then

$$T''(M, x^0, v) = cl\,\{cone\,[cone(M - x^0) - v]\} \text{ and } T^2(M, x^0, v) \subset T''(M, x^0, v).$$

See, e. g., Aubin and Frankowska (1990), Giorgi and Guerraggio (1992, 2002), Jimenez and Novo (2004), Giorgi, Jimenez and Novo (2010).

2 Optimality Conditions in Vector Optimization

In this Section we state some general optimality conditions (necessary and/or sufficient) for a vector optimization problem, using the first and second order tangent sets and cones previously introduced. We consider problem (1), where $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$, $M \subset \mathbb{R}^n$, D ordering cone (closed, convex, pointed and with $int(D) \neq \varnothing$) of $\mathbb{R}^p$.

Theorem 5.

Consider problem (1), where f is differentiable at x^0. If x^0 is a weak local efficient point for (1), then

$$\nabla f(x^0)y \notin -int(D), \forall y \in T(M, x^0). \tag{4}$$

Proof.

(Here $\nabla f(x^0)$ obviously denotes the Jacobian matrix of f at x^0). Let $y \in T(M, x^0)$; then there exist $\{x^n\} \subset M$, $\{\lambda_n\} \subset \mathbb{R}$, $x^n \longrightarrow x^0$, $\lambda_n \longrightarrow +\infty$, such that $\lambda_n(x^n - x^0) \longrightarrow y$. By Taylor's expansion we have

$$f(x^n) - f(x^0) = \nabla f(x^0)(x^n - x^0) + o(\| x^n - x^0 \|)$$

where $\lim\limits_{x^n \longrightarrow x^0} \frac{o(\|x^n - x^0\|)}{\|x^n - x^0\|} = 0$.

Consequently,

$$\lambda_n(f(x^n) - f(x^0)) = \nabla f(x^0)(\lambda_n(x^n - x^0)) + \lambda_n o(\| x^n - x^0 \|).$$

It results

$$\lambda_n o(\| x^n - x^0 \|) = \lambda_n \| x^n - x^0 \| \frac{o(\| x^n - x^0 \|)}{\| x^n - x^0 \|} \longrightarrow 0,$$

so that

$$\lambda_n(f(x^n) - f(x^0)) \longrightarrow \nabla f(x^0)y.$$

On the other hand the weak local efficiency of x^0 implies the existence of $\bar{n}$ such that $\forall n > \bar{n}$ it results

$f(x^n) - f(x^0) \notin -int(D)$, so that $\lambda_n(f(x^n) - f(x^0)) \notin -int(D)$ and thus

$$\lim_{x^n \longrightarrow x^0} \lambda_n(f(x^n) - f(x^0)) = \nabla f(x^0)y \notin -int(D).$$

The thesis follows. ■

We remark that if $p = 1,\ D = \mathbb{R}_+$ (scalar case), the optimality condition of Theorem 5 collapses to the well known necessary optimality condition: $\nabla f(x^0)y \geq 0, \forall y \in T(M, x^0)$, or equivalently $-\nabla f(x^0) \in (T(M, x^0))^*$, where A^* is the *polar cone* of A: $A^* = \{y \in \mathbb{R}^n : yx \leq 0, \forall x \in A\}$.

Obviously (4) is equivalent to

$$T(M, x^0) \cap C_0(f, x^0) = \varnothing, \tag{5}$$

where $C_0(f, x^0) = \{y \in \mathbb{R}^n : \nabla f(x^0)y \in -int(D)\}$.

The following theorem states a sufficient optmality condition for the same problem (1).

Theorem 6.

Consider problem (1), under the same assumptions of Theorem 5. A sufficient condition for x^0 to be a local efficient point for (1) is

$$\nabla f(x^0)y \notin -D, \forall y \in T(M, x^0), y \neq 0. \tag{6}$$

Proof.

Assume that x^0 is not a local efficient point for (1). Then there exists a feasible sequence $\{x^n\}$ with $x^n \longrightarrow x^0$ and $f(x^n) - f(x^0) \in -D \diagdown \{0\}$. By Taylor's expansion we have

$$\frac{f(x^n) - f(x^0)}{\| x^n - x^0 \|} = \nabla f(x^0) \left(\frac{x^n - x^0}{\| x^n - x^0 \|} + \frac{o(\| x^n - x^0 \|)}{\| x^n - x^0 \|} \right).$$

Since $\frac{x^n - x^0}{\|x^n - x^0\|} \longrightarrow y \in T(M, x^0)$ and $\frac{o(\|x^n - x^0\|)}{\|x^n - x^0\|} \longrightarrow 0$, we have $\nabla f(x^0)y \in -D$, in contradiction with the thesis. ■

If $M = \mathbb{R}^n$ or M is an open set or $x^0 \in int(M)$, problem (1) becomes an unconstrained problem. In such a case the tangent cone $T(M, x^0)$ is the whole space $\mathbb{R}^n$, so that the necessary optimality condition of Theorem 5 becomes

$$\nabla f(x^0)y \notin -int(D),\ \forall y \in \mathbb{R}^n, \tag{7}$$

whereas the sufficient optimality condition of Theorem 6 becomes

$$\nabla f(x^0)y \notin -D,\ \forall y \in \mathbb{R}^n,\ y \neq 0. \tag{8}$$

In the scalar case ($p = 1, D = \mathbb{R}_+$), (7) reduces to the classical Fermat rule $f'(x^0) = 0$, whereas (8) is inconsistent (unlike the vector case).

The optimality conditions (5) and (6) can be expressed by means of multipliers. Assume, for simplicity, that $D = \mathbb{R}^p_+$.

Theorem 7.

Consider the differentiable problem (1), where $M \subset \mathbb{R}^n$ is open or $x^0 \in int(M)$, and $D = \mathbb{R}^p_+$.

(i) If x^0 is a weak local efficient point, then

$$\exists \lambda \in \mathbb{R}^p_+ \setminus \{0\} : \lambda \nabla f(x^0) = 0. \tag{9}$$

(ii) If (9) holds with $\lambda \in int\,(\mathbb{R}^p_+)$, i.e. with λ *positive vector* of $\mathbb{R}^p$ and $\ker \nabla f(x^0) = \{0\}$, i.e. $rank \nabla f(x^0) = n$, then x^0 is a local efficient point for (1).

Proof.

(i) If we set $V = \{v \in \mathbb{R}^p : v = \nabla f(x^0)y,\ y \in \mathbb{R}^n\}$, the relation $\nabla f(x^0)y \notin -int\mathbb{R}^p_+,\ \forall y \in \mathbb{R}^n$, is obviously equivalent to state

$$V \cap -int\mathbb{R}^p_+ = \varnothing.$$

The thesis follows from a well known theorem of the alternative (Gordan theorem).

(ii) We can use the Stiemke theorem of the alternative: between the systems

$$\lambda \nabla f(x^0) = 0,\ \lambda > 0 \text{ and } \nabla f(x^0)v \geq 0,\ \nabla f(x^0)v \neq 0 \text{ or}$$
$$\nabla f(x^0)v \leq 0,\ \nabla f(x^0)v \neq 0,$$

one and only one has solutions. Taking into account that, being $rank \nabla f(x^0) = n$, the system $\nabla f(x^0)v = 0$ admits only the solution $v = 0$, we obtain the sufficient condition (8), with $D = \mathbb{R}^p_+$. ■

For a more general approach to sufficient first-order optimality conditions see Giorgi, Jimenez and Novo (2008). We have to note that condition (ii) is not very useful, as it is the same both for local efficient minimum points and for local efficient maximum points. In order to obtain useful conditions one has to impose some kind of generalized convexity (or concavity, in case of a maximum problem) on the objective function f (see, e. g., Cambini and Martein (1993, 1994)).

Now we give the efficiency criteria in the image space. Consider again problem (1), with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $M \subset \mathbb{R}^n, D$ closed convex pointed cone of $\mathbb{R}^p$, with $int(D) \neq \varnothing$. Denote by $E \subset \mathbb{R}^p$ the image of the feasible set, i. e. $E = f(M)$.

Theorem 8.

If $y^\circ \in E \subset \mathbb{R}^p$ is a local weak minimum of E (with respect to D), then the following conditions are satisfied:

(i) $T(E, y^\circ) \cap TI(-D, 0) = \varnothing$.

(ii) $T^2(E, y^\circ, u) \cap TI(-int(D), u) = \varnothing$, for all $u \in T(E, y^\circ) \cap bd(-D)$.

(iii) $T''(E, y^\circ, u) \cap TI(-int(D), u) = \varnothing$, for all $u \in T(E, y^\circ) \cap bd(-D)$.

Proof.
(i) From Theorem 2 (iv) we have that $TI(-D,0) = TI(-int(D),0)$. Let us suppose that there exists $v \in T(E,y^\circ) \cap TI(-D,0)$. As $v \in T(E,y^\circ)$, there exist sequences $\{y^n\} \subset E, \{y^n\} \longrightarrow y^\circ$ and $\{t_n\} \longrightarrow 0^+$, such that $v^n = ((y^n - y^\circ)/t_n) \longrightarrow v$, then

$$y^n = y^\circ + t_n v^n \in E, \ \forall n \in \mathbb{N}. \tag{10}$$

On the other hand, since $v \in TI(-int(D),0)$, there exists $\delta > 0$ such that $0 + tv' \in -int(D), \ \forall t \in (0,\delta], \forall v' \in N(v,\delta)$. Now, for this $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $t_n \in (0,\delta]$ and $v^n \in N(v,\delta)$ for all $n \geq n_0$. Making $t = t_n$ and $v' = v^n$, we have that $-d^n = t_n v^n \in -int(D), \ \forall n \geq n_0$ and taking (10) into account, it follows that

$$y^n = y^\circ - d^n \in E, \ d^n \in int(D)$$

in contradiction withthe local weak minimality of y°.
(ii) and (iii) The proofs of (ii) and (iii) are similar, taking the respective definitions of T^2 and T" into account, so we only prove part (iii). See also Jimenez and Novo (2004). Suppose that there exists $z \in T''(E,y^\circ,u) \cap TI(-int(D),u)$. By the definition of the set $T''(E,y^\circ,u)$, there exist sequences $(t_n,r_n) \longrightarrow (0^+,0^+)$ and $z^n \longrightarrow z$ such that $(t_n/r_n) \longrightarrow 0$ and

$$y^n = y^\circ + t_n u + \frac{1}{2} t_n r_n z^n \in E, \ \forall n \in \mathbb{N}. \tag{11}$$

On the other hand, as $z \in TI(-int(D),u)$, there exists $\delta > 0$ such that

$$u + \alpha z' \in -int(D), \forall \alpha \in (0,\delta), \ z' \in N(z,\delta).$$

For this $\delta > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\frac{1}{2} r_n \in (0,\delta)$ and $z^n \in N(z,\delta)$ for all $n \geq n_0$. So, $u + \frac{1}{2} r_n z^n \in -int(D)$, and consequently

$$-d^n = t_n u + \frac{1}{2} t_n r_n z^n \in -int(D).$$

Thus (11) can be written

$$y^n = y^\circ - d^n, \text{ with } d^n \in int(D),$$

in contradiction to the local weak efficiency of y°. ■

The theorem that follows (see Jimenez and Novo (2004)) establishes sufficient conditions for local efficiency in the image space of problem (1).

Theorem 9.

Let $y^\circ \in E$. If one of the following conditions holds:
(i) $T(E,y^\circ) \cap -D = \{0\}$.
(ii) For each $u \in T(E,y^\circ) \cap -D \setminus \{0\}$ we have

$$T^2(E,y^\circ,u) \cap u^\perp \cap -cl(cone(D+u)) = \varnothing$$

$$T''(E,y^\circ,u) \cap u^\perp \cap -cl(cone(D+u)) = \{0\}$$

then y° is a local efficient point of E.

Here $u^\perp$ denotes the orthogonal complement of u.

3 Optimality conditions under twice differentiability

In the present Section, as an application to vector optimization of second order tangent sets and asymptotic second order tangent cones, we assume twice Fréchet differentiability of the functions involved and shall establish, on the grounds of Jimenez and Novo (2004), second order necessary optimality conditions for problem (1) and second order sufficient conditions, without any "gap" between them.

We consider problem (1), with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $M \subset \mathbb{R}^n$, and the partial ordering in $\mathbb{R}^p$ defined through a closed, pointed convex cone D with nonempty interior. Let $x^0 \in M$ be a feasible point for problem (1). We continue to denote the Jacobian matrix of f at x^0 by $\nabla f(x^0)$, whereas $f''(x^0)$ denotes the second order Fréchet derivative of f at x^0 (in other words $f''(x^0)(v,v)$ is the vector whose $i-$th component is $v^\top \nabla^2 f_i(x^0)v$). Following Jimenez and Novo (2004), we define the following cones.

$C_0(f,x^0) = \{v \in \mathbb{R}^n : \nabla f(x^0)v \in -int(D)\}\,;$

$C(f,x^0) = \{v \in \mathbb{R}^n : \nabla f(x^0)v \in -D\}\,;$

Let $v \in C(f,x^0)$, then

$C_0^2(f,x^0,v) = \{w \in \mathbb{R}^n : \nabla f(x^0)w + f''(x^0)(v,v) \in -int(cone(D + \nabla f(x^0)v))\}\,,$

$C^2(f,x^0,v) = \{w \in \mathbb{R}^n : \nabla f(x^0)w + f''(x^0)(v,v) \in -cl(cone(D + \nabla f(x^0)v))\}\,,$

$C_0''(f,x^0,v) = \{w \in \mathbb{R}^n : \nabla f(x^0)w \in -int(cone(D + \nabla f(x^0)v))\}\,,$

$C''(f,x^0,v) = \{w \in \mathbb{R}^n : \nabla f(x^0)w \in -cl(cone(D + \nabla f(x^0)v))\}\,.$

Theorem 10.

Let x^0 be a local weak efficient point for (1). Then

(i) $T(M,x^0) \cap C_0(f,x^0) = \varnothing$.

(ii) For each $v \in T(M,x^0) \cap [C(f,x^0) \diagdown C_0(f,x^0)]$ it holds

$$T^2(M,x^0,v) \cap C_0^2(f,x^0,v) = \varnothing \tag{12}$$

$$T''(M,x^0,v) \cap C_0''(f,x^0,v) = \varnothing. \tag{13}$$

This proposition improves Theorem 3.1 in Bigi and Castellani (2000), Theorem 3.1 in Jimenez and Novo (2003), Theorem 3.7 in Cambini and Martein (2002) and Theorem 3.3 in Hachimi and Aghezzaf (2007). Obviously, part (i) is nothing but Theorem 5. Part (ii) is valid for all $v \in \mathbb{R}^n$, but is only meaningful for $v \in T(M,x^0) \cap [C(f,x^0) \diagdown C_0(f,x^0)]$. See Jimenez and Novo (2004).

If $D = \mathbb{R}^p_+$, we obtain the following necessary conditions for a Pareto problem.

Theorem 11.

Let in (1) be $D = \mathbb{R}^p_+$. If x^0 is a local weak efficient point for (1), then for each

$$v \in T(M,x^0) \cap \{v \in \mathbb{R}^n : \nabla f_i(x^0)v \leq 0, \quad \forall i = 1,2,...p \\ \text{and } \nabla f_i(x^0)v = 0 \text{ for some } i\}$$

the following systems in $w \in \mathbb{R}^n$ are incompatible:

$$(a) \qquad \begin{cases} w \in T^2(M, x^0, v) \\ \nabla f_i(x^0)w + f_i''(x^0)(v,v) < 0, \quad \forall i \in I(v), \end{cases}$$

$$(b) \qquad \begin{cases} w \in T''(M, x^0, v) \\ \nabla f_i(x^0)w < 0, \quad \forall i \in I(v), \end{cases}$$

where $I(v) = \{i : \nabla f_i(x^0)v = 0\}$.

The result is just Theorem 3.7 in Cambini and Martein (2002) and also coincides with Corollary 3.2 in Hachimi and Aghezzaf (2007).

Now we associate to the necessary conditions of Theorem 11 the sufficient conditions of the same type. We need a further definition and some previous results. Let us again consider problem (1), for the finite-dimensional case. The following notion was introduced by Jimenez (2002).

Definition 5.

Let $k \geq 1$ an integer. The point $x^0 \in M$ is said to be a strict local minimum of order k for problem (1), denoted $x^0 \in Strl(k, f, M)$, if there exist $\alpha > 0$ and a neighborhood U of x^0 such that

$$(f(x) + D) \cap N(f(x^0),\ \alpha\|x - x^0\|^k) = \varnothing,\ \forall x \in M \cap U \diagdown \{x^0\}.$$

When $D = \mathbb{R}^p_+$, such a point is called a strict local Pareto minimum of order k.

Definition 5 becomes the notion introduced by Hestenes (1966, 1975) of strict local minimum of order $k = 1, 2$, when $p = 1,\ D = \mathbb{R}_+$:

$$f(x) \geq f(x^0) + \alpha\|x - x^0\|^k,\ \forall x \in M \cap U \diagdown \{x^0\}.$$

We have that every strict local minimum of order k is also of order j, for $j \geq k$, and every strict local minimum of order k is a local minimum (see Jimenez (2002)).

We have the following results, due to Jimenez and Novo (2004), which show that by a joint use of T^2 and T'', there is no "gap" between necessary optimality conditions and sufficient optimality conditions for problem (1). In fact we have

$$\begin{aligned} T^2(M, x^0, v) \cap v^\perp \cap C_0^2(f, x^0, v) &= \varnothing \Longleftrightarrow T^2(M, x^0, v) \cap C_0^2(f, x^0, v) = \varnothing, \\ T''(M, x^0, v) \cap v^\perp \cap C_0''(f, x^0, v) &= \varnothing \Longleftrightarrow T''(M, x^0, v) \cap C_0''(f, x^0, v) = \varnothing. \end{aligned}$$

Theorem 12.

Let us consider problem (1) and let $x^0 \in M \subset \mathbb{R}^n$:

(i) $T(M, x^0) \cap C(f, x^0) = \{0\}$ if and only if $x^0 \in Strl(1, f, M)$.

(ii) If for every $v \in T(M, x^0) \cap [C(f, x^0) \diagdown \{0\}]$ we have

$$\begin{aligned} T^2(M, x^0, v) \cap v^\perp \cap C_0^2(f, x^0, v) &= \varnothing, \\ T''(M, x^0, v) \cap v^\perp \cap C_0''(f, x^0, v) &= \{0\}, \end{aligned}$$

then $x^0 \in Strl(2, f, M)$.

This proposition improves Theorem 3.8 in Cambini and Martein (2002), Theorem 3.6 in Hachimi and Aghezzaf (2007) and extends Theorem 4 in Cambini, Martein and Vlach (1999) to vector optimization. Moreover, relation (i) improves relation (i) of the previous Theorem 10. The inclusion of the orthogonal subspace to v in the sufficient conditions is a slight improvement, as it holds

$$T^2(M, x^0, v) \cap v^{\perp} \cap C^2(f, x^0, v) = \varnothing \Longleftrightarrow T^2(M, x^0, v) \cap C^2(f, x^0, v) = \varnothing$$

but

$$T''(M, x^0, v) \cap v^{\perp} \cap C''(f, x^0, v) = \{0\} \Longleftrightarrow T''(M, x^0, v) \cap C''(f, x^0, v) = lin\,\{v\}\,,$$

where $lin\,\{v\}$ is the subspace generated by v.

When $D = \mathbb{R}^p_+$ (Pareto case) we have the following sufficient optimality conditions for strict Pareto minimality.

Theorem 13.

Let in problem (1) be $D = \mathbb{R}^p_+,\ M \subset \mathbb{R}^n,\ f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$. If for each $v \in T(M, x^0) \diagdown \{0\}$ satisfying $\nabla f(x^0)v \leq 0$ one has that the following systems in $w \in \mathbb{R}^n$ are incompatible:

$$(a) \qquad \begin{cases} w \in T^2(M, x^0, v) \cap v^{\perp} \\ \nabla f_i(x^0)w + f_i''(x^0)(v, v) \leq 0,\ \forall i \in I(v), \end{cases}$$

$$(b) \qquad \begin{cases} w \in T''(M, x^0, v) \cap v^{\perp} \diagdown \{0\} \\ \nabla f_i(x^0)w < 0,\ \forall i \in I(v), \end{cases}$$

then x^0 is a strict local Pareto minimum of order 2 for f on M.

4 Optimality Conditions in Nonsmooth Vector Optimization Problems

In this Section we apply the previous results to a vector optimization problem with nonsmooth data, more precisely under Hadamard differentiability. We recall some basic definitions and properties.

Definition 6.

Let $f : X \longrightarrow \mathbb{R}$, with $X \subset \mathbb{R}^n$ and $x^0,\ v \in \mathbb{R}^n$.

(*a*) The *Hadamard derivative* of f at x^0 in the direction v is

$$df(x^0, v) = \lim_{(t,u) \longrightarrow (0^+, v)} \frac{f(x^0 + tu) - f(x^0)}{t}.$$

(*b*) The *directional derivative* (or *Dini derivative*) of f at x^0 in the direction v is

$$Df(x^0, v) = \lim_{t \longrightarrow 0^+} \frac{f(x^0 + tv) - f(x^0)}{t}.$$

(c) f is Hadamard (resp. Dini) differentiable at x^0 if there exists $df(x^0,v)$ (resp. $Df(x^0,v)$) for all $v \in X$. In the one-dimensional case ($\mathbb{R}^n = \mathbb{R}$) the Hadamard derivative coincides with the Dini derivative.

Definition 7.

A function $f : X \longrightarrow \mathbb{R}$, $X \subset \mathbb{R}^n$, is said to be *Lipschitz continuous* on a set $S \subset X$, *modulus* c ($c \geq 0$), if for all $x^1,\ x^2 \in S$ it follows that

$$| f(x^1) - f(x^2) | \leq c \parallel x^1 - x^2 \parallel .$$

If f is Lipschitz continuous (modulus c) in a neighborhood of a point $x^0 \in X$, f is said to be *locally Lipschitz* at x^0 (see, e. g. Clarke (1983)).

The above definitions can at once be generalized to vector-valued functions and to functions defined on a vector normed space and with values on another vector normed space. It is well known that:

$(\mathfrak{i})$ if f is Fréchet differentiable at x^0, then $\nabla f(x^0)v = df(x^0,v)$;

$(\mathfrak{ii})$ if there exists $df(x^0,v)$, then there exists $Df(x^0,v)$ and both derivatives are the same;

$(\mathfrak{iii})$ if f is locally Lipschitz at x^0 and there exists the Dini derivative at x^0, then there exists the Hadamard derivative at x^0;

$(\mathfrak{iv})$ if f is Hadamard differentiable at x^0, then f is continuous at x^0 and $df(x^0,.)$ is continuos on X. This property is not true for a Dini-type derivative.

See, e. g., Bonnans and Shapiro (2000), Demyanov and Rubinov (1995).

Next we recall the notion of Dini subdifferential.

Definition 8.

Let $f : X \longrightarrow \mathbb{R}$, $X \subset \mathbb{R}^n$, be Dini differentiable at x^0. The *Dini subdifferential* of f at x^0 is

$$\partial_D f(x^0) = \left\{ \xi \in \mathbb{R}^n : \xi v \leq Df(x^0,v),\ \forall v \in X \right\}.$$

It is well known that if $Df(x^0,.)$ is a convex function, then there exists the Dini subdifferential, i. e. this one is a nonempty set. If $Df(x^0,.)$ is not a convex function, then $\partial_D f(x^0)$ can be the empty set.

We consider again problem (1), with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $M \subset \mathbb{R}^n$, $D \subset \mathbb{R}^p$ (D closed, pointed convex cone with a nonempty interior) and f Hadamard differentiable at $x^0 \in M$. We redefine the critical cone $C(f,x^0)$ and the strict critical cone $C_0(f,x^0)$ as follows:

$$C^H(f,x^0) = \left\{ v \in \mathbb{R}^n : df(x^0,v) \in -D \right\}$$

$$C_0^H(f,x^0) = \left\{ v \in \mathbb{R}^n : df(x^0,v) \in -int(D) \right\}.$$

In the following lemma we prove an interesting property of the Hadamard derivative.

Lemma 1.

Let $M \subset \mathbb{R}^n$, $x^0 \in M$ and $v \in \mathbb{R}^n$.

(i) If there exist $df(x^0, v)$ and $v = \lim_{n \longrightarrow \infty} (x^n - x^0)/t_n$, with $t_n \longrightarrow 0^+$ and $x^n \in M$, then

$$\lim_{n \longrightarrow \infty} \frac{f(x^n) - f(x^0)}{t_n} = df(x^0, v).$$

(ii) If f is Hadamard differentiable at x^0, then

$$df(x^0, .)(T(M, x^0)) \subset T(f(M), f(x^0)).$$

Proof.

If $v \in T(M, x^0)$, then there exist sequences $\{x^n\} \longrightarrow x^0, \{x^n\} \subset M$, and $\{t_n\} \longrightarrow 0^+$ such that $v^n = ((x^n - x^0)/t_n) \longrightarrow v$, so

$$\begin{aligned} df(x^0, v) &= \lim_{(t,u) \longrightarrow (0^+, v)} \frac{f(x^0 + tu) - f(x^0)}{t} = \lim_{n \longrightarrow \infty} \frac{f(x^0 + t_n v^n) - f(x^0)}{t_n} \\ &= \lim_{n \longrightarrow \infty} \frac{f(x^n) - f(x^0)}{t_n} \end{aligned}$$

and part (i) follows.

Since f is continuous at x^0 we have that $\{f(x^n)\} \longrightarrow f(x^0)$. Taking into account that $f(x^n) \in f(M)$ and that $\{t_n\} \longrightarrow 0^+$, we deduce that

$$df(x^0, v) \in T(f(M), f(x^0)),$$

and the proof is finished. ∎

As an application of Theorem 8 we prove the following first order necessary optimality conditions for problem (1), under Hadamard differentiability.

Theorem 14.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be Hadamard differentiable at $x^0 \in M$. If x^0 is a local weak efficient point for problem (1), then

$$T(M, x^0) \cap C_0^H(f, x^0) = \varnothing,$$

i.e.

$$df(x^0, v) \notin -int(D), \forall v \in T(M, x^0).$$

Proof.

If x^0 is a local weak efficient point for (1), there exists a neighborhood of x^0 such that $f(x^0) = y^\circ$, with y° local weak efficient vector for the set $f(M \cap U)$, always under the ordering expressed by the cone D. From Theorem 8 we have that

$$T(f(M \cap U), y^\circ) \cap TI(-D, 0) = \varnothing,$$

equivalent to

$$T(f(M \cap U), y^\circ) \cap (-int(D)) = \varnothing.$$

Since $T(M \cap U, x^0) = T(M, x^0)$, from Lemma 1 it follows that

$$df(x^0, .)(T(M, x^0)) = df(x^0, .)(T(M \cap U, x^0)) \subset T(f(M \cap U), y^\circ),$$

consequently $df(x^0, .)(T(M, x^0)) \cap (-int(D)) = \varnothing$. Using the inverse of $df(x^0, .)$, we conclude that $T(M, x^0) \cap C_0^H(f, x^0) = \varnothing$. ■

Similarly to Theorem 6, a sufficient local optimality condition can be stated for problem (1), under Hadamard differentiability: a sufficient condition for $x^0 \in M$ to be a local efficient point for problem (1) is:

$$T(M, x^0) \cap C_0^H(f, x^0) = \{0\}\,, \tag{14}$$

i. e.

$$df(x^0, v) \notin -D, \forall v \in T(M, x^0), v \neq 0. \tag{15}$$

However, this result can be improved; indeed, Jimenez and Novo (2004) have proved the following proposition.

Theorem 15.

Let us suppose that $x^0 \in M \subset \mathbb{R}^n$ and f Hadamard differentiable at x^0. Then (14) (or equivalently (15)) holds if and only if $x^0 \in Strl(1, f, M)$.

Obviously, Theorems 14 and 15 hold also under the assumption that f is Dini differentiable at $x^0 \in M$ and locally Lipschitz at x^0. In the scalar case ($p = 1, D = \mathbb{R}_+$), Theorem 14 gives the necessary optimality condition

$$df(x^0, v) \geq 0,\ \forall v \in T(M, x^0),$$

whereas Theorem 15 gives the sufficient optimality condition

$$df(x^0, v) > 0,\ \forall v \in T(M, x^0),\ v \neq 0,$$

which is also necessary for a strict local minimum of order one. These last results appear in the literature from time to time, in a less or more general formulation. See, e. g., Ben Tal and Zowe (1985), L. Qi (2001), Huang (2005).

5 A Multiplier Rule for a Nonsmooth Multiobjective Pareto Programming Problem

Several multiplier rules for problems (2) and (3) have been proposed by various authors, making use of first order tangent cones, second order tangent sets and second order asymptotic cones. The literature is quite abundant; we quote only the books of Ehrgott (2005), Jahn (2005), Luc (1989), Miettinen (1999), Sawaragi, Nakayama and Tanino (1985).

We quote also the papers of Giorgi, Jimenez and Novo (2004a, 2004b) and of Jimenez and Novo (2002a, 2002b, 2003, 2004, 2008).

Here we consider the following Pareto program with equality and inequality constraints

$$Min\ f(x),\ \text{subject to } x \in S \tag{16}$$

where

$$S = \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\},$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}^r$, i. e. we consider problem (3), defined on real finite-dimensional spaces and with the ordering cones given by $\mathbb{R}^p_+$ and $\mathbb{R}^m_+$ and without the set constraint Q. Problem (16) is the classical Pareto optimization problem. We denote by $f_i,\ i \in I = \{1, 2, ..., p\}$, $g_j,\ j \in J = \{1, 2, ..., m\}$, $h_k,\ k \in K = \{1, 2, ..., r\}$, the component functions of f, g and h, respectively. The set of active indices of g at the feasible point x^0 is

$$J_0 = \left\{j \in J : g_j(x^0) = 0\right\}.$$

Then we denote $G = \{x \in \mathbb{R}^n : g(x) \leq 0\}$, $H = \{x \in \mathbb{R}^n : h(x) = 0\}$, therefore $S = G \cap H$. We assume the following two conditions:

(H1) f and g are Hadamard differentiable with convex derivatives.

(H2) h is Fréchet differentiable and $\nabla h(x^0)$ has maximal rank, i. e. the vectors $\nabla h_k(x^0),\ k \in K$, are linearly independent.

We define the strict critical cones formed by the objective function and by the inequality constraints:

$$\begin{aligned} C_0^H(f, x^0) &= \{v \in \mathbb{R}^n : df(x^0, v) \in -int(\mathbb{R}^p_+)\} = \\ &= \{v \in \mathbb{R}^n : df_i(x^0, v) < 0,\ \forall i \in I\}, \\ C_0^H(G, x^0) &= \{v \in \mathbb{R}^n : dg_j(x^0, v) < 0,\ \forall j \in J\}. \end{aligned}$$

We need two previous results.

Theorem 16. (Jimenez and Novo (2002b)).

Under the assumptions (H1) and (H2) we have

$$C_0^H(G, x^0) \cap \ker(\nabla h(x^0)) \subset T(S, x^0).$$

We recall that a proper function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *sublinear* if

$$\begin{aligned} f(x+y) &\leq f(x) + f(y),\ \forall x, y \in \mathbb{R}^n \\ f(\alpha x) &= \alpha f(x),\ \forall x \in \mathbb{R}^n,\ \forall \alpha > 0, \end{aligned}$$

i. e. a proper sublinear function is a convex function positively homogeneous of first degree. We recall also the classical notion of *subdifferential* of a convex function f. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex, the set

$$\partial f(x^0) = \left\{v \in \mathbb{R}^n : f(y) \geq f(x^0) + (y - x^0)v,\ \forall y \in \mathbb{R}^n\right\}$$

is the subdifferential of f at x^0. Moreover, for convex functions (defined on open convex subsets of $\mathbb{R}^n$) the subdifferential of f at x^0 can be characterized by Dini derivatives:

$$\partial f(x^0) = \partial_D f(x^0) = \left\{ v \in \mathbb{R}^n : vy \leq Df(x^0, y),\ \forall y \in \mathbb{R}^n \right\}.$$

Theorem 17. (Jimenez and Novo (2002a)).

Let us suppose that $\varphi_1, \varphi_2, ..., \varphi_q : \mathbb{R}^n \longrightarrow \mathbb{R}$ are sublinear functions and $\psi_1, \psi_2, ..., \psi_r : \mathbb{R}^n \longrightarrow \mathbb{R}$ are linear functions given by $\psi_k(u) = c^k u,\ k \in K = \{1, 2, ..., r\}$. Then one and only one of the following assertions are true:

(a) There exists $v \in \mathbb{R}^n$ such that

$$\begin{cases} \varphi_i(x^0, v) < 0,\ \forall i = 1, 2, ...q \\ \psi_k(v) = 0,\ \forall k = 1, 2, ...k. \end{cases}$$

(b) There exists $(\xi, \upsilon) = (\xi_1, \xi_2, ...\xi_q, \upsilon_1, \upsilon_2, ...\upsilon_k) \in \mathbb{R}^{q+r},\ \xi \neq 0,\ \xi \geq 0$, such that

$$0 \in \sum_{i=1}^{q} \xi_i \partial \varphi_i(0) + \sum_{k=1}^{r} \upsilon_k c^k.$$

Theorem 18.

Let us consider the Pareto programming problem (16) and let us assume conditions (H1) and (H2). If x^0 is a local efficient point for (16), then there exists $(\lambda, \mu, \upsilon) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that

$$(\lambda, \mu) \geq 0,\ (\lambda, \mu) \neq 0 \tag{17}$$

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_D f_i(x^0) + \sum_{j=1}^{m} \mu_j \partial_D g_j(x^0) + \sum_{k=1}^{r} \upsilon_k \nabla h_k(x^0) \tag{18}$$

$$\mu_j g_j(x^0) = 0,\ j = 1, ..., m. \tag{19}$$

If, in addition, $C_0^H(S, x^0) \neq \varnothing$, then $\lambda \neq 0$.

Proof.

As x^0 is a local weak efficient point for (16), from Theorem 14 we have that

$$T(S, x^0) \cap C_0^H(f, x^0) = \varnothing, \tag{20}$$

but, since in this case $C_0^H(f, x^0) = \{v \in \mathbb{R}^n : df_i(x^0, v) < 0, \forall i \in I\}$, condition (20) means that there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} df_i(x^0, v) < 0,\ \forall i \in I \\ v \in T(S, x^0). \end{cases} \tag{21}$$

Now, from Theorem 16 we have that

$$C_0^H(G, x^0) \cap \ker(\nabla h(x^0)) \subset T(S, x^0).$$

So, taking (20) into account, there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} df_i(x^0, v) < 0, \ \forall i \in I \\ dg_j(x^0, v) < 0, \ \forall j \in J_0 \\ \nabla h_k(x^0)v = 0, \ \forall k \in K, \end{cases} \tag{22}$$

and using Theorem 17 the conclusion follows, choosing $\mu_j = 0$ for all $j \in J \diagdown J_0$.

For the second part of the theorem, let us suppose that $C_0^H(S, x^0) \neq \varnothing$, that is, there exists $w \in \mathbb{R}^n$ such that

$$dg_j(x^0, w) < 0, \ \forall j \in J_0, \ \nabla h_k(x^0)w = 0, \ \forall k \in K. \tag{23}$$

Assume that $\lambda = 0$. Then conditions (17)-(19) imply that

$$\sum_{j \in J_0} \mu_j dg_j(x^0, u) + \sum_{k=1}^{r} \upsilon_k \nabla h_k(x^0)u \geq 0, \forall u \in \mathbb{R}^n$$

with $\mu \neq 0$. For $u = w$, we have a contradiction, since from (23) it follows that

$$\sum_{j \in J_0} \mu_j dg_j(x^0, w) + \sum_{k=1}^{r} \upsilon_k \nabla h_k(x^0)w < 0.$$

Consequently $\lambda \neq 0$. ■

6 On the Use of the Guignard-Gould-Tolle Constraint Qualification in Vector Optimization Problems.

In discussing a gap between multiobjective optimization and scalar optimization (a gap first pointed out by Wang and Yang (1991)), Aghezzaf and Hachimi (2001) state that "in multiobjective optimization problems, many authors have derived the first-order and second-order necessary conditions under the Abadie constraint qualification, but never under the Guignard constraint qualification". This deserves a clarification. Several authors have proposed a suitable Guignard-Gould-Tolle constraint qualification to obtain a Karush-Kuhn-Tucker type multiplier rule for a Pareto optimization problem. For example, Maeda (1994) considers the following Pareto optimization problem

$$Min \ f(x), \ \text{subject to } g(x) \leq 0,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p, \ g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and introduces the following "generalized Guignard constraint qualification" for this problem:

$$C(Q, x^0) \subset \cap_{i=1}^{p} cl(conv(T(Q^i, x^0))),$$

where x^0 is a feasible vector,

$$\begin{aligned} Q &= \left\{x \in \mathbb{R}^n : g(x) \le 0,\ f(x) \le f(x^0)\right\} \\ Q^i &= \left\{x \in \mathbb{R}^n : g(x) \le 0,\ f_k(x) \le f_k(x^0),\ k = 1, 2, ...p \text{ and } k \ne i\right\} \\ C(Q, x^0) &= \left\{h \in \mathbb{R} : \nabla f_i(x^0)h \le 0,\ i = 1, ..., p;\ \nabla g_j(x^0)h \le 0,\ j \in I(x^0)\right\}. \end{aligned}$$

($I(x^0) = \{i : g_i(x^0) = 0\}$). Indeed, the inclusion in the above constraint qualification means in fact equality. Similarly, Jimenez and Novo (1999), Giorgi, Jimenez and Novo (2004a, 2009), Giorgi and Zuccotti (2011); in this last paper there is a misprint: the equality sign between the first and the second member of the Guignard constraint qualification has been omitted.

What is true is that the Guignard-Gould -Tolle theory cannot be transferred "sic et simpliciter" from the scalar to the vector case. It is well known that if we have a scalar optimization problem

$$Min\ f(x),\ \text{subject to } x \in M \subset \mathbb{R}^n,$$

with $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, f at least differentiable at x^0, if x^0 is a local solution of the above problem, then

$$-\nabla f(x^0) \in (T(M, x^0))^*. \tag{24}$$

The result obtained by Guignard (1969) seems more general, as Guignard claims that

$$-\nabla f(x^0) \in (P(M, x^0))^*,$$

where $P(M, x^0) = cl(conv(T(M, x^0)))$ is the so-called *pseudotangent cone* to M at x^0. However, this greater generality is only apparent, as it is true that for any cone C, it holds $C^* = (cl(conv(C)))^*$, so we obtain $(T(M, x^0))^* = (P(M, x^0))^*$.

Relation (24) obviously is equivalent to the inconsistency of

$$\nabla f(x^0)y < 0 \text{ for } y \in T(M, x^0), \text{ or equivalently for } y \in P(M, x^0).$$

We have seen in Theorem 5 that if x^0 is a local weak efficient point for (1), with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ differentiable, $M \subset \mathbb{R}^n$, $D = \mathbb{R}^p_+$, then we have the relation

$$\nabla f(x^0)v \notin -int(\mathbb{R}^p_+),\ \forall v \in T(M, x^0),$$

i. e. the system

$$\nabla f_i(x^0)v < 0,\ i = 1, ..., p,$$

has no solution for $v \in T(M, x^0)$. One may wonder if this last system (for $p > 1$) is also inconsistent for $v \in (conv(T(M, x^0)))$, as it holds for $p = 1$. The answer is: no, as shown by Wang and Yang (1991) with a numerical example (a misprint in this example has been corrected by Castellani and Pappalardo (2001)). This is the "gap" between scalar and vector optimization problems, with reference to a result of Guignard, to which the paper of Wang and Yang (1991) makes reference. Sufficient conditions to remove this "gap" are:

(a) the cone $T(M, x^0)$ is convex;

(b) the objective function f is *subconvexlike* on its domain $X \subset \mathbb{R}^n$, i. e. for any $x^1, x^2 \in X$, $\lambda \in (0,1)$ and $a \in int(\mathbb{R}^p_+)$, there exists $x^3 \in X$ such that

$$a + \lambda f(x^1) + (1-\lambda) f(x^2) - f(x^3) \in \mathbb{R}^p_+.$$

See, for further considerations, Castellani and Pappalardo (2001).

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