Quaderni di Dipartimento

A simple financial market model with chartists and fundamentalists: market entry levels and discontinuities

Fabio Tramontana
(Università di Pavia)

Frank Westerhoff
(University of Bamberg)

Laura Gardini
(Università di Urbino)

A simple financial market model
with chartists and fundamentalists:
market entry levels and discontinuities

Fabio Tramontana
Department of Economics and Quantitative Methods,
University of Pavia, Italy, fabio.tramontana@unipv.it
Frank Westerhoff
Department of Economics, University of Bamberg, Germany
frank.westerhoff@uni-bamberg.de
Laura Gardini
Department of Economics, Society and Politics,
University of Urbino, Italy, laura.gardini@uniurb.it

Abstract

We present a simple financial market model with interacting chartists and fundamentalists. Since some of these speculators only become active when a certain misalignment level has been crossed, the dynamics are driven by a discontinuous piecewise linear map. The model endogenously generates bubbles and crashes and excess volatility for a broad range of parameter values – and thus explains some key phenomena of financial markets. Moreover, we provide a complete analytical study of the model's dynamical system. One of its surprising features is that model simulations may appear to be chaotic, although only regular dynamics can emerge.

Keywords: financial market crisis; bull and bear market dynamics; discontinuous piecewise linear maps; border-collision bifurcations; period adding scheme.

1 Introduction

Our paper seeks to add to the burgeoning literature on agent-based financial market models which explain the dynamics of financial markets by highlighting the trading activity of their participants. Seminal contributions in this field include Day and Huang (1990), Chiarella (1992), de Grauwe et al. (1993), Kirman (1993), Lux (1995), Brock and Hommes (1998), LeBaron et al. (1999), Farmer and Joshi (2002) and He and Li (2008). According to this class of models, interactions between heterogeneous and boundedly rational speculators, relying on simple technical and fundamental trading rules, can generate complex endogenous price dynamics, including, for instance, the emergence of bubbles and crashes. More recent approaches are surveyed in Hommes (2006), LeBaron (2006), Lux (2009), Chiarella et al. (2009) and Westerhoff (2009).

A few papers in this exciting area focus on the dynamics of piecewise linear maps. Such piecewise linear maps, which may be regarded as an approximation

of more complicated nonlinear maps, have the advantage that they often allow for a deeper analytical study of the underlying dynamical system, and thus advance our understanding of what is driving the dynamics of financial markets. For examples, see the asset pricing models of Huang and Day (1993), Day (1997), Huang et al. (2010) and Tramontana et al. (2010).

Our model, representing a stylized speculative market with interacting chartists and fundamentalists, also has a piecewise linear structure[1]. The reason for this is that we assume that while some speculators are always active in the market, others only become active when a certain misalignment level has been crossed. Since we assume otherwise linear technical and fundamental trading rules, the model consists of three disconnected branches. The inner regime is due to the transactions of speculators who are always active; the two outer regimes depend on the joint trading behavior of all market participants.

From a mathematical point of view, the peculiarity of our model is that although numerically we can observe trajectories that may look chaotic, chaotic behavior cannot occur. Instead, only regular dynamics are possible, as the trajectories are either periodic or quasiperiodic. However, both cases are structurally unstable, as they are never persistent under a parameter variation. It should also be noted that discontinuous piecewise linear maps have not yet been thoroughly studied. Despite their simplicity, they can, however, lead to surprising new insights. We hope that our paper will advance our knowledge of such maps.

From an economic point of view, our model explains, at least partially, the excess volatility and the disconnect puzzle – which are two of the most challenging puzzles in international finance (see, e.g. Shiller 2005). We find this rather interesting since the only assumption required for this is that, in an otherwise linear world, there are different market entry levels for certain types of speculator. This assumption, which appears quite natural to us, is already sufficient for creating endogenous price dynamics.

After these introductory remarks, the plan of the paper is as follows. In section 2, we introduce our model and describe some preliminary properties of its underlying dynamical system. In section 3, we start to investigate the model in more detail. Since different parameter assumptions yield different maps, the analysis stretches over sections 3 to 5. In section 6, we summarize and prove two main results. Finally, section 7 concludes the paper.

2   A discontinuous financial market model

Overall, our model consists of rather standard building blocks, formalizing the behavior of a market maker, and four types of speculator. However, a special feature is that we assume the so-called type 1 chartists and type 1 fundamentalists always active in the market, whereas the so-called type 2 chartists and

---

[1]Note that there is abundant empirical evidence, summarized by Menkhoff and Taylor (2007), which supports the view that speculators indeed rely on technical and fundamental trading rules.

type 2 fundamentalists only become active when prices deviate at least a certain minimum amount from fundamentals. Of course, the more attention a market receives, the more extreme its prices are. On the one hand, this may trigger an additional inflow of chartists who perceive an exploitable bubble. On the other hand, there may also be an additional inflow of fundamentalists who believe they can profit from a fundamental price correction.

Since our simple model concentrates on transactions of heterogeneous speculators, it can, with some liberty, be seen as a stylized representation of a stock, commodity or foreign exchange market. The model will be presented in section 2.1. In section 2.2, we will then discuss some properties of our piecewise linear maps, and related maps, which are helpful for understanding and appreciate the properties of our model.

2.1 Our model's building block

The first building block of our model describes price adjustments. Following Day and Huang (1990), we assume a market maker mediates transactions out of equilibrium by providing or absorbing liquidity, depending on whether the excess demand is positive or negative. In addition to clearing the market, the market maker quotes prices according to the following rule

$$P_{t+1} = P_t + a \left(D_t^{C,1} + D_t^{F,1} + D_t^{C,2} + D_t^{F,2} \right), \tag{1}$$

where P is the price, a is a positive price adjustment parameter, and $D_t^{C,1}$, $D_t^{F,1}$, $D_t^{C,2}$ and $D_t^{F,2}$ are the orders of the four types of speculator. Accordingly, excess buying drives the price up and excess selling drives it down. For simplicity, yet without loss of generality, we set scaling parameter a equal to 1.

Chartists believe in the persistence of bull and bear markets. The orders of type 1 chartists are therefore given by

$$D_t^{C,1} = c^1 \left(P_t - F \right), \tag{2}$$

where c^1 is a positive reaction parameter and F stands for the asset's (constant) fundamental value. Hence type 1 chartists submit buying orders in bull markets and selling orders in bear markets[2].

The trading behavior of fundamentalists is exactly contrary to the trading behavior of chartists. We formalize the orders of type 1 fundamentalists by

$$D_t^{F,1} = f^1 \left(F - P_t \right), \tag{3}$$

where f^1 is a positive reaction parameter. Clearly, (3) generates buying orders when the market is overvalued and generates selling orders when it is undervalued.

---

[2]This building block also goes back to Day and Huang (1990). Note that Boswijk et al. (2007) and Westerhoff and Franke (2011) report empirical support for such kind of trading behavior.

What type 1 chartists and type 1 fundamentalists have in common is that they are (almost) always active. Once they perceive mispricing, they start trading. Type 2 chartists and type 2 fundamentalists are different to them in the sense that they only become active when the misalignment exceeds a certain critical threshold level. The orders of type 2 chartists and type 2 fundamentalists are therefore represented by

$$D_t^{C,2} = \begin{cases} 0 & if \quad |P_t - F| < z \\ c^2 \left(P_t - F\right) & if \quad |P_t - F| > z \end{cases} \tag{4}$$

and

$$D_t^{F,2} = \begin{cases} 0 & if \quad |P_t - F| < z \\ f^2 \left(F - P_t\right) & if \quad |P_t - F| > z \end{cases} \tag{5}$$

respectively. Again, reaction parameters c^2 and f^2 are positive and the aforementioned threshold level is given by $z > 0$.

It is convenient to express the model in terms of deviations from its fundamental value. Using auxiliary variable $X_t = P_t - F$ and combining (1) to (5) yields

$$X_{t+1} = \begin{cases} (1 + c^1 - f^1)X_t & if \quad |X| < z \\ (1 + c^1 - f^1 + c^2 - f^2)X_t & if \quad |X| > z \end{cases}, \tag{6}$$

which is a one-dimensional map consisting of three linear, disconnected straight lines.

Furthermore, it is useful to introduce definitions $S^1 = c^1 - f^1$ and $S^2 = c^2 - f^2$. Note first that S^1 and S^2 can take any values. A positive (negative) value of S^1 means that type 1 chartists are more (less) aggressive than type 1 fundamentalists. Of course, the same interpretation holds for S^2 and type 2 speculators: a positive (negative) value of S^2 now means that type 2 chartists are more (less) aggressive than type 2 fundamentalists.

At first sight, it might appear peculiar that type 2 chartists and type 2 fundamentalists become active simultaneously when the distance between the price and the fundamental value becomes larger than z and, indeed, a more general model might allow for two different threshold levels (which would result in a map with five linear branches). However, in an even simpler version of our model we can have any positive value for S^2 if we assume that there are only type 2 chartists and any negative value for S^2 if we assume that there are only type 2 fundamentalists. As we shall see later on, the latter specification, implying additional fundamentalists, is particularly interesting (and economically quite reasonable). For the moment, however, we shall stick to the more general setup which includes both type 2 chartists and type 2 fundamentalists.

To simplify the notation even further, let us write $X' = X_{t+1}$ and $X = X_t$. Then (6) can be expressed as

$$\overline{F}: \quad X' = \begin{cases} (1 + S^1)X & if \quad |X| < z \\ (1 + S^1 + S^2)X & if \quad |X| > z \end{cases}. \tag{7}$$

This is the map we explore in detail in the rest of the paper.

4

2.2 Some preliminary properties

First, however, it is helpful to contrast some properties of map (7) with those of the following map

$$F: \quad X' = \begin{cases} (1 + S^1)X + E & if \quad |X| < z \\ (1 + S^1 + S^2)X & if \quad |X| > z \end{cases}, \tag{8}$$

where the extra parameter E can be positive or negative[3].

A first property is that parameter z is a scale variable. In fact, by using the change of variable $x = X/z$ and defining the aggregate parameter $M = E/z$, our model in (8) becomes

$$F: \quad x' = \begin{cases} (1 + S^1)x + M & if \quad |x| < 1 \\ (1 + S^1 + S^2)x & if \quad |x| > 1 \end{cases}. \tag{9}$$

That is, we have the following

Property 1. *The map in (8) is topologically conjugated to the map in (9).*

Note that M can be positive, negative or zero. However, the two cases with a positive and negative sign of M are topologically conjugated to one another. We have the following

Property 2. *The map F in (9) with $M < 0$ is topologically conjugated with the same map F and $M > 0$.*

In fact, by using the change of variable $y = -x$, the map in *(9)* leads to

$$F: \quad y' = \begin{cases} (1 + S^1)y - M & if \quad |y| < 1 \\ (1 + S^1 + S^2)y & if \quad |y| > 1 \end{cases}. \tag{10}$$

Clearly, the property holds also for map F in (8) with the sign of E. Hence, model (9) can be expressed as:

$$F: \quad x' = \begin{cases} g(x) = (1 + S^1 + S^2)x & if & x < -1 \\ f(x) = (1 + S^1)x + M & if & -1 < x < 1 \\ g(x) = (1 + S^1 + S^2)x & if & x > 1 \end{cases}, \tag{11}$$

and is represented by a one-dimensional piecewise linear discontinuous map, with two discontinuity points.

Investigating dynamics of this kind of map is quite new, and not yet fully understood. We can therefore have some generic dynamic properties for our class of maps, which are related to the piecewise linear structure. As we shall see, the case with $M = 0$ is very special. The numerical simulations of the observed dynamics may lead to incorrect conclusions, reflecting a sequence of states very close to chaotic behavior, although no chaos can occur. In fact, this case leads to a non-chaotic map with peculiar properties, with regular dynamics,

---

[3]Note that map (8) with $E \neq 0$ corresponds to a financial market model which is studied in Tramontana et al. (2011).

either being periodic or quasiperiodic, and will be completely investigated in this paper.

By contrast, when $M \neq 0$, the dynamic behavior generally includes attracting cycles (structurally stable, as persistent for variation of each parameter in some interval) or truly chaotic dynamics (also structurally stable or robust, i.e. persistent under parameter variation). The most important property for these piecewise linear maps is that the appearance of cycles cannot occur via a fold (or tangent) bifurcation, as is usual in smooth maps. Instead, a cycle can appear/disappear only via a *border collision bifurcation*. This term, initially used in papers by Nusse and Yorke (1992, 1995), is now used extensively in the literature of piecewise smooth systems. A cycle undergoes a border collision bifurcation when one of its periodic points merges with a discontinuity point.

Even if map (9) can generate cycles with periodic points in two or three of its partitions, there are only two functions involved, so that the eigenvalue of a cycle depends only on the number of periodic points in which functions $f(x)$ and $g(x)$ are applied. Moreover, the flip bifurcations are not the usual ones (we recall that for smooth maps it is associated with the appearance of a stable cycle of double period). In piecewise linear maps only *degenerate flip bifurcations* can occur, so that at the bifurcation value a whole segment of cycles of double period exists, stable but not asymptotically stable. The dynamic effects, after the bifurcation, are not uniquely defined. It is possible to have several kinds of dynamics, but often this bifurcation leads to chaotic sets, that is to cyclic chaotic intervals (see Sushko and Gardini, 2010). Thus the following property holds:

Property 3. *A structurally stable cycle of map F in (9) can appear/disappear only via a border collision bifurcation. The eigenvalue of a cycle having p periodic points in the middle region ($|x| < 1$) and q outside ($|x| > 1$) is given by $\lambda = (1 + S^1)^p (1 + S^1 + S^2)^q$. Only degenerate-type flip bifurcations can occur.*

Moreover, another property of map (9) is also immediate, and excludes cases which are unfeasible in the applied context, as leading to divergent trajectories. We know from property 3 that when both slopes of functions $f(x)$ and $g(x)$ are in modulus higher than 1, then all of the possible cycles are unstable, as $|\lambda| > 1$. In these cases, a piecewise linear map can only have chaotic dynamics (when bounded trajectories exist) or divergent trajectories. However, due to the particular structure of our map, when $|1 + S^1| > 1$ and $|1 + S^1 + S^2| > 1$, we cannot have bounded dynamics because function $g(x)$ is linear. This implies that whatever the dynamics in the range $|x| < 1$, where the map is affine, in a finite number of iterations any not fixed trajectory enters the region with $|x| > 1$, where it depends on the iterations of an expanding linear function ($g(x)$, the graph of which is through the origin). The length of the interval bounded by 0 and x_t can therefore only increase at each step. The unique possible existing cycle is thus an unstable fixed point. Hence we have proved the following

Property 4. *Consider map F in (9) with $|1 + S^1| > 1$ and $|1 + S^1 + S^2| > 1$. Then any initial condition different to the unstable fixed point (if existing) has a divergent trajectory.*

Economically, $|1+S^1| > 1$ means either that type 1 chartists are slightly more aggressive than type 1 fundamentalists ($S^1 > 0$) or that type 1 fundamentalists are considerably more aggressive than type 1 chartists ($S^1 < -2$). Moreover, $|1 + S^1 + S^2| > 1$ may be interpreted in the sense that the joint impact of type 1 and type 2 chartists dominates, at least slightly, over the joint impact of type 1 and type 2 fundamentalists or that the joint impact of type 1 and type 2 fundamentalists is much stronger than the joint impact of type 1 and type 2 chartists. We learn from this, furthermore, that not only chartists but also fundamentalists can contribute to market instability.

In the statement of property 3 we considered structurally stable cycles, which can occur only for $M \neq 0$. Depending on the values of the parameters, as such positive or negative slopes of functions f and g, we can have different dynamic properties. The possible outcomes associated with $M \neq 0$ will be investigated in a different paper, while the dynamics existing when $M = 0$ is the object of the present study. As already remarked, the case $M = 0$ is special as only structurally unstable dynamics, either periodic or quasiperiodic, can exist.

3 Non-chaotic regime at $M = 0$

Let us consider map F in (11), for the particular case $M = 0$, say F_0 (which corresponds to map $\overline{F}$ in (7) after the change of variable $x = X/z$):

$$F_0: \quad x' = \begin{cases} f(x) = (1 + S^1)x & if \quad |x| < 1 \\ g(x) = (1 + S^1 + S^2)x & if \quad |x| > 1 \end{cases} \tag{12}$$

and keeping all of the possible values for the slopes of functions $f(x)$ and $g(x)$, that is $(1 + S^1)$ and $(1 + S^1 + S^2)$ can be positive or negative and in modulus higher or smaller than 1. We can therefore consider the regions in the parameter space (S^1, S^2), as summarized in Fig. 1.

Before proceeding to comment on behavior in the parameter space, let us remark on one further property specific to this case $M = 0$, which holds in the phase space of variable x. Performing the change of variable $y = -x$, the map is transformed into itself:

$$y' = \begin{cases} f(y) = (1 + S^1)y & if \quad |y| < 1 \\ g(y) = (1 + S^1 + S^2)by & if \quad |y| > 1 \end{cases} \tag{13}$$

which means that the phase space is symmetric with respect to the origin. That is: either a trajectory is symmetric with respect to the origin or the symmetric one also exists. This is particularly true for a periodic orbit. We have therefore proved the following

Property 5. *Map F_0 is invariant with respect to the change of variable $y = -x$. Thus a periodic orbit $(x_1, x_2, ...x_n)$ either has points symmetric with respect to the origin or $(-x_1, -x_2, ... - x_n)$ is also a periodic orbit.*
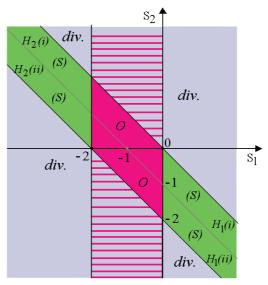
Fig. 1 Two-dimensional parameter space (S^1, S^2) at $M = 0$. The regions are bounded by straight lines $S^1 = 0$, $S^1 = -2$, $S^2 = -S^1$, $S^2 = -S^1 - 2$. Line $S^2 = -S^1 - 1$ leads only to a qualitative change.

Let us now consider the parameter space. In Fig. 1 the regions with divergent dynamics are those already introduced in Property 4; those associated with the stability of the fixed point in origin $O = (0, 0)$ are described in the following

Property 6. *Consider map F_0 with $|1 + S^1| < 1$. For $|1 + S^1 + S^2| < 1$ fixed point O in the origin is globally attracting. For $|1 + S^1 + S^2| > 1$ fixed point O is attracting, with basin of attraction $\mathcal{B}(O) =]-1, 1[$, while any i.c. x with $|x| > 1$ has a divergent trajectory.*

In fact, if $|1 + S^1 + S^2| < 1$, then any initial condition in the range $|x| > 1$ has a trajectory which, in a few iterations, enters range $|x| < 1$ from which the trajectory converges to the origin. This leads to the red region in Fig. 1, while the dynamics in the other regions of the vertical strip of Fig. 1 are associated with $|1 + S^1 + S^2| > 1$. In such a case, any initial condition in the range $|x| < 1$ has a trajectory which converges to the origin, as it is locally stable and the map is linear in that region, while any initial condition in range $|x| > 1$, due to the structure of the piecewise linear map, has a trajectory which is divergent.□

Similar to before, these cases can be interpreted economically. For instance, the unique fixed point of the model, where the price is equal to its fundamental value, is globally stable if type 1 fundamentalists are more aggressive than type 1 chartists, but also not too aggressive ($-2 < S^1 < 0$) and also if the joint impact of both types of fundamentalists is stronger, yet not very much stronger ($S^1 + S^2$ has to remain larger than -2), than the joint impact of both types of chartist.

The particular cases with $(1 + S^1) = 1$ and $(1 + S^1) = -1$, that is $S^1 = 0$ and $S^1 = -2$, are *degenerate bifurcations* (as described in Sushko and Gardini

(2010)). For $S^1 = 0$ there is segment $]-1,1[$ filled with fixed points; for $S^1 = -2$ segment $]-1,1[$ is filled with period 2 cycles. At these degenerate bifurcations, the existing cycles are stable but not asymptotically stable (i.e. they do not attract the trajectories of nearby points). After the bifurcation, for $|1 + S^1| > 1$, the result depends on the modulus of $(1 + S^1 + S^2)$. As we have seen, for $|1 + S^1 + S^2| > 1$ only divergent dynamics can occur, while for $|1 + S^1 + S^2| < 1$ an invariant absorbing interval J exists, given by:

$$
\begin{aligned}
J &= [f(-1), f(1)] = [-(1+S^1), (1+S^1)], \quad \text{if } (1+S^1) > 1 \qquad (14) \\
J &= [f(1), f(-1)] = [(1+S^1), -(1+S^1)], \quad \text{if } (1+S^1) < -1,
\end{aligned}
$$

attracting the trajectories of all points of the phase space outside J (and from which a trajectory cannot escape). Thus the dynamics cannot be divergent.

It follows that the particular cases left to our analysis are exactly those in the green regions of Fig. 1, which is the main object of our work. As visible from Fig. 1, the regions under investigation are really four different regions, associated with different values of the slopes of functions $f(x)$ and $g(x)$. For $S^1 > 0$, these include the two cases

$$
\begin{aligned}
H_1(i) &: \quad (1+S^1) > 1, \ 0 < (1+S^1+S^2) < 1, \ \text{increasing/increasing} \quad (15) \\
H_1(ii) &: \quad (1+S^1) > 1, \ -1 < (1+S^1+S^2) < 0, \ \text{increasing/decreasing},
\end{aligned}
$$

while for $S^1 < -2$, these include the two cases

$$
\begin{aligned}
H_2(i) &: \quad (1+S^1) < -1, \ 0 < (1+S^1+S^2) < 1, \ \text{decreasing/increasing} \quad (16) \\
H_2(ii) &: \quad (1+S^1) < -1, \ -1 < (1+S^1+S^2) < 0, \ \text{decreasing/decreasing}.
\end{aligned}
$$

Again, these four regions have a simple economic interpretation. For instance, case $H_1(i)$ states that type 1 chartists are more aggressive than type 1 fundamentalists, but that the joint impact of both types of fundamentalist is stronger than the joint impact of both types of chartist. The difference between case $H_1(i)$ and case $H_1(ii)$ is that the joint impact of both types of fundamentalist is stronger in case $H_1(ii)$, yet also not too much stronger ($S^1 + S^2$ has to remain above -2). Obviously, the main difference between the two H_1 cases and the two H_2 cases is then that the H_2 cases imply that type 1 fundamentalists are so aggressive that they destabilize the steady state within the inner regime. Global stability will, however, still be maintained as long as $|1 + S^1 + S^2| < 1$. Given the assumption $S^1 < -2$, it is then clear that aggressive type 2 chartists are required to prevent price explosions.

In the next sections, we shall fully explain cases $H_1(i)$ and $H_1(ii)$, which will also be used to explain cases H_2. Let us first introduce the peculiar property of our model described by map F_0, which is stated in the following

Property (S). *Consider map F_0 with $|1 + S^1| > 1$ and $|1 + S^1 + S^2| < 1$. Then the following equalities hold*:

$$
(S): \ f \circ g(1) = g \circ f(1) \ , \ \ f \circ g(-1) = g \circ f(-1). \qquad (17)
$$

9

In fact, this property can be immediately verified from the definition of map F_0 given in (12): we have $g \circ f(1) = (1 + S^1 + S^2)(1 + S^1)$ and $f \circ g(1) = (1 + S^1)(1 + S^1 + S^2)$ as well as $g \circ f(-1) = -(1 + S^1 + S^2)(1 + S^1)$ and $f \circ g(-1) = -(1 + S^1)(1 + S^1 + S^2)$, so that the properties in (17) hold.□

Property (S) is an important property because it leads to a *stability regime* which is, however, structurally unstable, that is: any small change in any parameter of the model leads to a different dynamic behavior. The important dynamic property of map F in this case $M = 0$ is exactly this Property (S) which, as we shall see, *implies that an invariant set I exists, and each point of I has a unique rank-1 preimage in the set I itself.* This property (that each point of I has a unique rank-1 preimage in the set I itself) is exactly the property of a linear rotation on a circle and, depending on a suitable rotation number, which in our case is associated with the values of parameters S^1 and S^2, a trajectory may be either periodic (in which case all of the points of the interval I are periodic of the same period), or quasiperiodic and dense in the interval I. In case $H_1(i)$ (increasing/increasing), considered in the next section, there are two disjoint invariant absorbing intervals: I^R and I^L. In case $H_1(ii)$ (increasing/decreasing), considered thereafter, the invariant set I will be the union of two intervals.

Let us analyze the conditions leading to periodic dynamics. Let x be a point belonging to the absorbing set I of map F_0, different to a discontinuity point. Then it can be a periodic point of first period n if n is the minimum integer such that $F_0^n(x) = x$. Let p be the number of periodic points of the $n-$cycle in the region $|x| < 1$ and q in the region $|x| > 1$, $(p + q) = n$. Then we have

$$F_0^n(x) = (1 + S^1)^p (1 + S^1 + S^2)^q x. \tag{18}$$

It follows that the condition of periodic orbit, $(1 + S^1)^p (1 + S^1 + S^2)^q x = x$, can be satisfied by a point $x \neq 0$ iff the eigenvalue $\lambda = (1 + S^1)^p (1 + S^1 + S^2)^q$ of the cycle satisfies the following equation

$$(1 + S^1)^p (1 + S^1 + S^2)^q = 1, \tag{19}$$

and thus the eigenvalue is $\lambda = 1$. We have so proved the following

Property 7. *Consider map F_0 with $|1 + S^1| > 1$ and $|1 + S^1 + S^2| < 1$. Then x is a periodic point of an $n-$cycle iff (19) holds, where p is the number of periodic points of the $n-$cycle in region $|x| < 1$ and q in the region $|x| > 1$, with $(p + q) = n$, and the eigenvalue of the cycle is $\lambda = (1 + S^1)^p (1 + S^1 + S^2)^q = 1$.*

On the other hand, the fact that the eigenvalue is equal to 1 means that the cycle is stable but not attracting, and in the piecewise linear case this can only occur for all points of an interval. That is, map F_0 necessarily satisfies condition $F_0^n(x) = x$ for all points x of a suitable interval, invariant for F_0^n, all points of which are periodic of the same period and with the same symbol sequence (i.e. with the same sequence of applied functions $f(x)$ and $g(x)$). Examples shall be given in the following sections, where the different cases are considered.

4 Dynamics in case $H_1(i)$, increasing/increasing

Let us consider here the effects of Property (S) for the dynamics when the map has the two functions $f(x)$ and $g(x)$, both with positive slopes $(1 + S^1) > 1$ and $0 < (1 + S^1 + S^2) < 1$, as qualitatively shown in Fig. 2a.

Under such assumptions, the map leads to two coexisting absorbing intervals, and thus *we necessarily have bistability*. In fact, any initial condition in region $x > 0$ will forever be in that region, entering the absorbing interval $I^R = [g(1), f(1)]$ in a finite number of iterations, from which it cannot escape. Thus it attracts the points in $\mathcal{B}(I^R) =]0, +\infty[$, which is its basin of attraction. The restriction of map F_0 to absorbing interval I^R is given by

$$F^R: \quad x' = \begin{cases} f(x) = (1 + S^1)x & if \quad g(1) < x < 1 \\ g(x) = (1 + S^1 + S^2)x & if \quad 1 < x < f(1) \end{cases}, \quad (20)$$

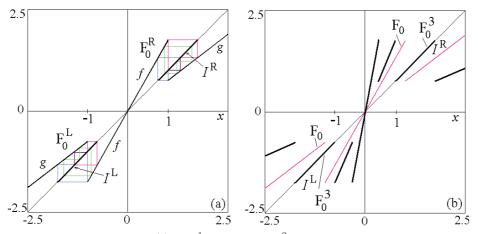where $g(1) = (1 + S^1 + S^2) \in (0, 1)$ and $f(1) = (1 + S^1) > 1$.

Fig. 2 Map F_0 in case $H_1(i)$ at $S^1 = 0.75$ and $S^2 = -0.9940711$ is shown in (a).

Similarly, any initial condition in region $x < 0$ will forever be in that region, entering the absorbing interval $I^L = [f(-1), g(-1)]$ in a finite number of iterations, from which it cannot escape, and it attracts the points in $\mathcal{B}(I^L) =]-\infty, 0[$. The restriction of map F_0 to absorbing interval I^L is given by

$$F^L: \quad x' = \begin{cases} g(x) = (1 + S^1 + S^2)x & if \quad f(-1) < x < -1 \\ f(x) = (1 + S^1)x & if \quad -1 < x < g(-1) \end{cases}, \quad (21)$$

where $f(-1) = -(1 + S^1) < -1$ and $g(-1) = -(1 + S^1 + S^2) \in (-1, 0)$.

Which kind of dynamics, then, can we have inside the two invariant absorbing intervals? Since no divergent trajectory can occur, we can argue that an initial condition in the intervals leads to some attracting set. However, this is not the case. An attracting set (or attractor) is defined as some invariant set for which a neighborhood exists whose points converge to the attractor. But

this cannot occur in our map, due to the existence of property (S). In fact, it is known (as shown in Keener 1980, see also Gardini et al. 2010) that in the case of a piecewise smooth increasing discontinuous map, the property in (17) leads to a map which is conjugated with a linear rotation. This means that, depending on the values of S^1 and S^2, a suitable rotation number may be defined, which may be rational or irrational. For a rational rotation number, all points of absorbing intervals $I^{R/L}$ are periodic (and all of the same period). For an irrational rotation number, all points of absorbing intervals $I^{R/L}$ have quasiperiodic trajectories dense in absorbing intervals $I^{R/L}$, but are not chaotic. Thus no true attracting set can exist, but the dynamics are regular: when there are periodic orbits, these are stable but not attracting. This is also the case when there are quasiperiodic trajectories. Moreover, these dynamics are structurally unstable, as they depend on a rational or irrational rotation number, which cannot persist when varying the parameters.

An example of periodic orbits is shown in Fig. 2 for case $H_1(i)$ at $S^1 = 0.75$ and $S^2 = -0.9940711$ (the reason why this value arises is explained below). At these parameter values, all points of invariant intervals I^R and I^L are periodic of period 3 (see Fig. 2a). The third iterate of the map is shown in Fig. 2b. It consists in several branches, one of which belongs to the diagonal on invariant interval I^R, and a second branch on the diagonal on interval I^L.

The main result for our map is that *this dynamic property is always true, independent of the values of the slopes,* in the regions marked with (S) in Fig. 1. That is, for map F_0 in which we are interested, this kind of non-chaotic regime, characterized by structurally unstable orbits (either periodic or quasiperiodic), is persistent for both parameters in cases $H_1(i$ and $ii)$ and cases $H_2(i$ and $ii)$, previously defined.

Let us consider here a few more properties on the organization of the existing cycles. Property 7, in the previous section, states when a cycle can exist. However, is it possible to find the exact values of p and q that give us the cycles? And is it possible to somehow organize their existence regions (which are curves in the two-dimensional parameter plane (S^1, S^2))?

In case $H_1(i)$, we can follow the same technique used in the case of attracting cycles when the so-called period adding scheme works. Indeed, as shown in Gardini et al. 2010, the intersection of the existing periodicity regions with the locus (S) of the stable (but not attracting) regime where Property (S) holds is a set of points in the locus which still follows the adding mechanism. We can therefore reason similarly in our case. It is clear that, in order to have the sequence of a so-called maximal cycle in interval I^R, say with symbol sequence fg^k, we have to look for a periodic point that can be obtained as a fixed point of composite function $g^k \circ f(x)$, solving of the equation $g^k \circ f(x) = x$. For their existence we have to determine all parameters S^1 and S^2 which satisfy, for any $k \geq 1$,

$$fg^k : \quad (1 + S^1)(1 + S^1 + S^2)^k = 1. \tag{22}$$

Thus we have curves in the parameter plane (S^1, S^2) given by:

$$S^2 = -(1 + S^1) + \frac{1}{(1 + S^1)^{1/k}}, \tag{23}$$

a few of which (for $k = 1, ..., 10$) are drawn in Fig. 3a. For $k = 2$ we have the $3-$cycles. For $S^1 = 0.75$, therefore, we have computed from (23) the value $S^2 = -0.9940711$, used to draw the example in Fig. 2.

Following the adding mechanism, we can find two families of infinite curves associated with cycles of second level of complexity between any two consecutive curves associated with maximal cycles, or cycles of first level of complexity. For example, we have the following pair of families of infinite curves (both for any $m \geq 1$) between the two curves fg^k and fg^{k+1} :

$$(fg^k)^m fg^{k+1} \quad : \quad (1 + S^1)^{1+m}(1 + S^1 + S^2)^{k+1+mk} = 1 \tag{24}$$
$$: \quad S^2 = -(1 + S^1) + 1/(1 + S^1)^{(1+m)/(k+1+mk)}$$

$$fg^k(fg^{k+1})^m \quad : \quad (1 + S^1)^{1+m}(1 + S^1 + S^2)^{k+m(1+k)} = 1 \tag{25}$$
$$: \quad S^2 = -(1 + S^1) + 1/(1 + S^1)^{(1+m)/(k+m+mk)}.$$

A few of these curves are shown in Fig. 3b for $k = 1, ..., 10$ and $m = 1, 2, 3$. In Fig. 3c the curves of Fig. 3a,b are shown together (inside each pair of green curves of Fig. 3a we have those in blue and red from equations (24) and (25)).

Similarly, we can continue for any level of complexity: between any two consecutive curves, with symbol sequence A and B, of the same level of complexity, we can compute two families of infinitely many curves, with symbol sequence $(A)^n B$ and $A(B)^n$, for any $n \geq 1$.

Exchanging f and g, we obtain a maximal cycle existing in interval I^R, with different symbol sequence, gf^k. A periodic point can be obtained as a fixed point of function $f^k \circ g(x)$. We therefore have to determine all parameters S^1 and S^2 such that, for any $k \geq 1$:

$$gf^k : \quad (1 + S^1 + S^2)(1 + S^1)^k = 1 \tag{26}$$

and two families of curves of cycles of second complexity level are given, for any $m \geq 1$, by:

$$gf^k(gf^{k+1})^m \quad : \quad (1 + S^1 + S^2)^{1+m}(1 + S^1)^{k+m(1+k)} = 1 \tag{27}$$
$$: \quad S^2 = -(1 + S^1) + 1/(1 + S^1)^{(k+m+mk)/(1+m)}$$

$$(gf^k)^m gf^{k+1} \quad : \quad (1 + S^1 + S^2)^{1+m}(1 + S^1)^{k+1+mk} = 1 \tag{28}$$
$$: \quad S^2 = -(1 + S^1) + 1/(1 + S^1)^{(k+1+mk)/(1+m)}$$

and so on for any level. A few of the curves in (26) are drawn in region (i) in Fig. 3d for $k = 1, ..., 10$. In Fig. 3e the curves from equations (27) and (28)

are drawn for $k = 1, ..., 10$ and $m = 1, 2, 3$; in Fig.3f the curves of Fig.3 d,e are shown together.

Under assumption $H_1(i)$, the infinitely many curves for which parameters (S^1, S^2) are associated with periodic orbits are dense in that region. However, if we numerically compute a bifurcation diagram, we observe a figure as shown in Fig. 4, where variable x is reported as a function of S^2 at $S^1 = 0.75$ fixed.
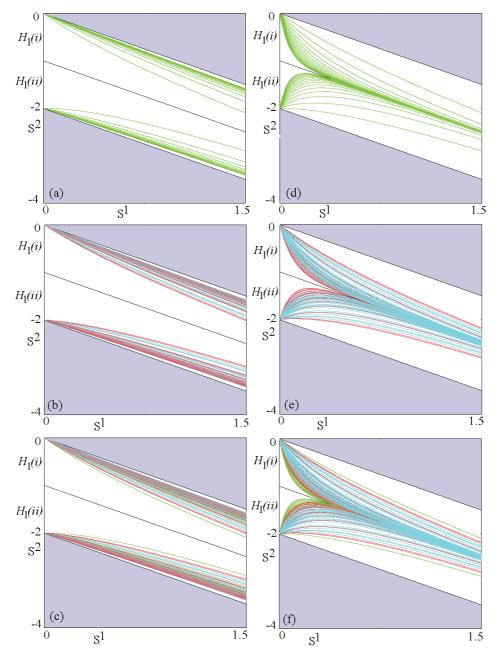
Fig. 3 Curves drawn analytically in regions $H_1(i)$ and $H_1(ii)$, as explained in the text, associated with periodic orbits of first and second complexity level.

Fig. 4 One-dimensional bifurcation diagram for map F_0 showing x as a function of S^2 in both regions $H_1(i)$ and $H_1(ii)$.

For $S^1 = 0.75$ fixed, the region corresponding to assumption $H_1(i)$ is the interval $-1.75 < S^2 < -0.75$. There we have two disjoint and coexisting invariant absorbing intervals I^R (in black in Fig. 4) and I^L (in red in Fig. 4). The numerical results are qualitatively similar to those which can be obtained in a chaotic regime. However, no chaotic regime can exist here. Since there are either periodic points or quasiperiodic trajectories at all the parameters values, and due to the fact that both the values of periodic orbits and quasiperiodic orbits are dense in the interval, we can numerically observe mainly a quasiperiodic orbit.
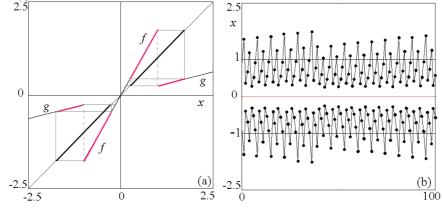

Fig. 5 Map F_0 in case $H_1(i)$ at $S^1 = 0.75$ and $S^2 = -1.5$ is shown in (a). (b) shows versus time behavior of two coexisting trajectories at the same parameters as in (a).

We notice that the versus time trajectory may also be misleading. It may be considered chaotic, although this cannot be the case. An example is shown in

16

Fig. 5. Fig. 5b shows the versus time behaviors of two coexisting trajectories, one in absorbing interval I^R and the other in absorbing interval I^L.

5 The other regions

Fig. 3 also shows the curves described above, drawn reflected in region $H_1(ii)$. That is, if parameter $(\overline{S^1}, \overline{S^2})$ belongs to a curve in region $H_1(i)$, then also the parameter which is symmetric with respect to curve $S^2 = -(1+S^1)$ necessarily belongs to a curve in region $H_1(ii)$ associated with a periodic orbit of F_0.

We can also generalize the reasoning, saying that all curves existing in region $H_1(i, ii)$ with $(1 + S^1) > 1$ must also have the symmetric curves in region $(1 + S^1) < -1$, in $H_2(i, ii)$, associated with periodic orbits.

To show this, let us define slopes $a = (1 + S^1)$ and $b = (1 + S^1 + S^2)$. Then let us consider parameters $(\overline{a}, \overline{b})$ corresponding to a point $(\overline{S^1}, \overline{S^2})$ belonging to a curve in region $H_1(i)$. Then also parameters $(\overline{a}, -\overline{b})$ necessarily belong to a curve associated with a periodic orbit of F_0, in region $H_1(ii)$. In fact, we know that

$$\overline{a}^p \overline{b}^q = 1 \tag{29}$$

for some suitable integers p and q. Then *if q is even, we also have*

$$\overline{a}^p (-\overline{b})^q = 1, \tag{30}$$

in which case the symmetric curve is associated with *a cycle of the same period* ($n = p + q$). Otherwise, *if q is odd, we have $\overline{a}^p (-\overline{b})^q = -1$ and*

$$\overline{a}^{2p} (-\overline{b})^{2q} = 1, \tag{31}$$

which means that the symmetric curve corresponds to *a cycle of double period* ($2n = 2(p + q)$).

For example, in the case of the 3-cycle shown in Fig. 2 at $(a, b) = (1.75, 0.7559289)$, we have $p = 1$ and $q = 2$, which is even. Thus we must also have 3-cycles at $(a, -b) = (1.75, -0.7559289)$, corresponding to a curve in region $H_1(ii)$, as is in fact shown in Fig. 6.

To parameters $(a, b) = (1.75, 0.829826534)$ corresponds a curve in the region $H_1(i)$ (from (22) with $k = 3$). The region is associated with 4-cycles with $p = 1$ and $q = 3$ which is odd (see Fig.7a), and it follows that at $(a, -b) = (1.75, -0.829826534)$ corresponds a curve in the region $H_1(ii)$ and we must have 8-cycles, as is in fact shown in Fig. 7b.
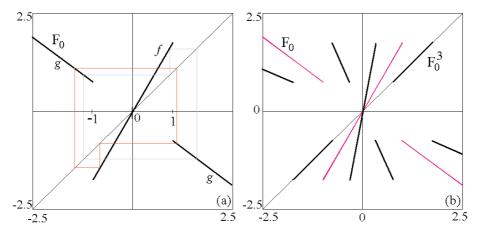
Fig. 6 Map F_0 in case $H_1(ii)$ at $a = (1 + S^1) = 1.75$ and
$b = (1 + S^1 + S^2) = -0.7559289$. In (a) all points are periodic of period 3. (b) also
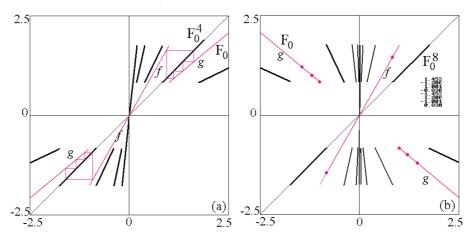shows map F_0^3 which has two intervals on the diagonal.

Fig. 7 Map F_0 in case $H_1(i)$ at $a = (1 + S^1) = 1.75$ and
$b = (1 + S^1 + S^2) = 0.829826534$ with 4-cycles and F_0^4 are shown in (a). In (b) at
parameters $(a, -b)$, corresponding to a point in $H_1(ii)$, there exist all 8-cycles, and
F_0^8 is shown.

Similarly, *if p is even, we also have*

$$(-\overline{a})^p \overline{b}^q = 1, \tag{32}$$

in which case the symmetric curve in region $H_2(i)$ is associated with *a cycle of
the same period* $(n = p + q)$. Otherwise, *if p is odd we have* $(-\overline{a})^p \overline{b}^q = -1$ *and*

$$(-\overline{a})^{2p} \overline{b}^{2q} = 1, \tag{33}$$

which means that the symmetric curve in region $H_2(i)$ corresponds to *a cycle
of double period* $(2n = 2(p + q))$.

18

While considering the symmetric point in region $H_2(ii)$, we necessarily have

$$(-\bar{a})^p(-\bar{b})^q = 1 \tag{34}$$

when p and q are *both even or both odd*, in which case we have *cycles of the same period*, and when p and q are *one odd and one even*, from $(-\bar{a})^p(-\bar{b})^q = -1$, then we have

$$(-\bar{a})^{2p}(-\bar{b})^{2q} = 1, \tag{35}$$

in which case it corresponds to *cycles of double period*.

An example is shown in Fig. 8. Considering the 4-cycle in Fig. 7a, at $(a, b) = (1.75, 0.829826534)$, belonging to a curve in region $H_1(i)$ associated with 4-cycles with $p = 1$ and $q = 3$, at $(-a, b) = (-1.75, 0.829826534)$, corresponding to a point in $H_2(i)$, we must have 8-cycles, as is in fact shown in Fig. 8a. At $(-a, -b) = (-1.75, -0.829826534)$, corresponding to a point in $H_2(ii)$, we must have 4-cycles, as shown in Fig. 8b.
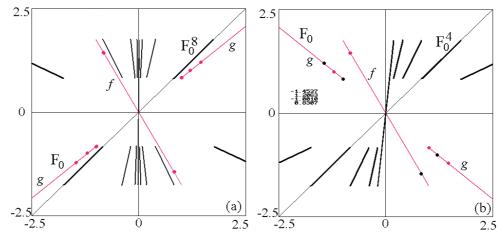

Fig. 8 Map F_0 in case $(-a, b) = (-1.75, 0.829826534)$, corresponding to a point in $H_2(i)$, shows all 8-cycles in (a), as well as F_0^8. In (b), in the case $(-a, -b) = (-1.75, -0.829826534)$, corresponding to a point in $H_2(ii)$, we have all 4-cycles, and F_0^4 is shown.

It is clear from the remarks given here that the curves associated with periodic orbits existing in region $H_1(i)$ (where the curves are dense) also exist in all other regions. Further details will be given in the next subsections.

5.1 Dynamics in case $H_1(ii)$, increasing/decreasing

While the results associated with the case under assumptions $H_1(i)$ has already been proved in the literature, we do not have similar results for case $H_1(ii)$ (nor for $H_2(i, ii)$). However, as already suggested in the previous section, the dynamics are exactly the same as those described in the increasing/increasing

case, and we can analytically write the curves for which we can find all periodic orbits and of any level of complexity. Let us consider here the parameters which satisfy conditions $H_1(ii)$. Then the property in (17) still holds, meaning that even if the map has increasing and decreasing branches (see Fig. 9), it is uniquely invertible in the invariant absorbing set, given by

$$I = [f(-1), g(1)] \cup [f(1), g(-1)]. \tag{36}$$

As a consequence of Property (S), in set I the map has either all periodic points dense in I or quasiperiodic trajectories dense in I. A numerically obtained bifurcation diagram is shown in Fig. 4 at $S^1 = 0.75$ fixed, in the region corresponding to assumption $H_1(ii)$, which is the interval $-2.75 < S^2 < -1.75$. Although the figure suggests chaotic behavior, it is not. We can determine the curves associated with periodic orbits. In fact, regarding the structure of the existing cycles, we can see that in this case, in a periodic orbit function $g(x)$ is necessarily applied an even number of times. Thus on the curves of region $H_1(ii)$, which are symmetric of those of region $H_1(i)$, either the period is the same (if the number of applications of g is even (i.e. if q is even in $\overline{a}^p \overline{b}^q = \overline{(1 + S^1)}^p \overline{(1 + S^1 + S^2)}^q = 1$) or it corresponds to a cycle of double period (as already shown in the previous section).
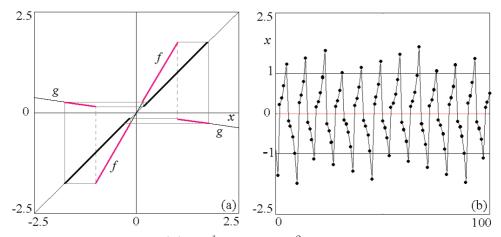

Fig. 9 Map F_0 in case $H_1(ii)$ at $S^1 = 0.75$ and $S^2 = -1.9$ is shown in (a). (b) shows versus time trajectories of x at the same parameter values as in (a), in the absorbing interval I defined in (36).

5.2 Dynamics in cases $H_2(i, ii)$

The dynamics in the case of assumptions $H_2(i, ii)$ are similar, since they can be reduced to those of cases $H_1(i, ii)$ using the second iterate of the map. In fact, let us consider values $(1 + S^1) < -1$. Then a rank-1 preimage of the discontinuity points $x = 1$ and $x = -1$ exists for the function in the range

20

$|x| < 1$. Explicitly, the preimages are given by

$$-1 < x_l = \frac{1}{(1+S^1)} < 0, \quad 0 < x_r = \frac{-1}{(1+S^1)} < 1,$$

satisfying $f(x_l) = 1$ and $f(x_r) = -1$. Clearly, these two points are discontinuity points for the second iteration $(F_0)^2$ of the map which, in interval $x_l < x < x_r$, is defined by a linear increasing function (i.e. with positive slope):

$$f^2(x) = (1+S^1)^2 x.$$

Then considering case $H_2(i)$, for the second iteration $(F_0)^2$ on the right and left side of interval (x_l, x_r), we have to apply f once and g once, so that the result is a negative sloped function defined outside interval $x_l < x < x_r$. In fact, the main point is that the second iterate $(F_0)^2$ is a continuous function in points $x = 1$ and $x = -1$, as we can immediately verify by direct computation or, stated differently, as a consequence of Property (S). This leads us to a map $(F_0)^2$, which is topologically conjugated to the case increasing/decreasing already considered in case $H_1(ii)$, with discontinuity points in x_l and x_r in place of -1 and 1, respectively, and slopes given by $(1+S^1)^2 > 0$ and $(1+S^1)(1+S^1+S^2) < 0$ in place of $(1+S^1)$ and $(1+S^1+S^2)$, respectively.

An example is shown in Fig. 10a. However, we notice that although the versus time dynamics of map $(F_0)^2$ when non periodic is qualitatively similar to that in Fig. 9b, the versus time dynamics of map F_0 is different, as shown in Fig. 10b.
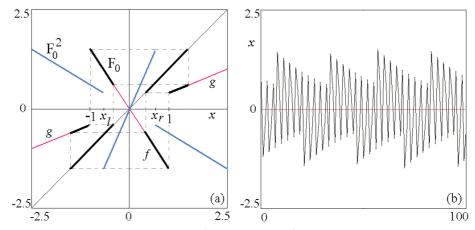

Fig. 10 Map F_0 in case $H_2(i)$ at $S^1 = -2.5$ and $S^2 = 1.9$s shown in (a). (b) shows versus time trajectories of x at the same parameter values as in (a), in the absorbing interval I.

By contrast, considering case $H_2(ii)$, for the second iteration $(F_0)^2$ nothing changes in interval (x_l, x_r), where the function is $f^2(x)$. While when now we apply, outside that interval, f once and g once, the result is a positive sloped

function, and the second iterate $(F_0)^2$ is a continuous function in the points $x = 1$ and $x = -1$, as we can immediately verify by direct computation, or as a consequence of Property (S). Function $(F_0)^2$ is therefore now topologically conjugated to that already considered in case $H_1(i)$, with discontinuity points in x_l and x_r in place of -1 and 1, respectively, and slopes given by $(1+S^1)^2 > 0$ and $(1+S^1)(1+S^1+S^2) > 0$ in place of $(1+S^1)$ and $(1+S^1+S^2)$, respectively. That is, we have two coexisting invariant absorbing intervals, one in region $x > 0$ and the other in region $x < 0$, see Fig.11a. However, for map F_0 there is always a unique absorbing interval I, and the states jump from the positive region to the negative one, and *vice versa*. An example of the versus time trajectory is shown in Fig. 11b.
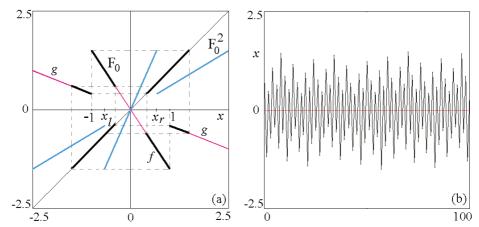

Fig. 11 Map F_0 in case $H_2(ii)$ at $S^1 = -2.5$ and $S^2 = 1.1$ is shown in (a). (b) shows versus time trajectories of x at the same parameter values as in (a), in the absorbing interval I.

6   Main results

Our main results are summarized in this section.

Property 8. *Consider map F_0 with $|1 + S^1| > 1$ and $|1 + S^1 + S^2| < 1$, then all of the trajectories enter an invariant absorbing interval I, inside which we can have either all periodic orbits or all quasiperiodic trajectories.*

Proof.

Under our assumptions, the dynamics of the map cannot be divergent, and the asymptotic dynamics are confined in invariant absorbing intervals. In case $H_1(i)$ we have two invariant absorbing intervals $I^R = [g(1), f(1)]$ and $I^L = [f(-1), g(-1)]$; in case $H_1(ii)$ the invariant absorbing interval is $I = [f(-1), g(1)] \cup [f(1), g(-1)]$. Case $H_2(i, ii)$ is reduced to $H_1(i, ii)$ for the second iterate of the map. We recall that in a piecewise-linear map, the $\omega-$limit set of a point x (i.e. the limit set of the trajectory of a point x) can only be a periodic

orbit or a quasiperiodic orbit dense in some intervals or cyclical chaotic intervals (see Sharkovsky et al. 1997). Our map shows that Property (S) is satisfied, from which we know that each point of the invariant absorbing interval has only one rank-1 preimage in the interval itself, *thus chaos cannot exist* (since noninvertibility is as a necessary condition for chaos). It follows that as $\omega-$limit set we can have only periodic orbits or quasiperiodic orbits. As for any point x in the invariant absorbing interval, we have

$$F_0^n(x) = (1 + S^1)^p (1 + S^1 + S^2)^q x,$$

where p is the number of points in region $|x| < 1$, q is the number of points in region $|x| > 1$ and $n = p + q$. It follows that when a suitable pair of integers p and q satisfy $(1 + S^1)^p (1 + S^1 + S^2)^q = 1$, then x is a periodic point of period n, with eigenvalue 1, and this must necessarily occur for all points x of suitable intervals (due to the linearity of the components). When integers p and q are such that this does not occur, then the trajectories are quasiperiodic.□

We then know from Property 7 when periodic orbits can occur. We have also learned from the discussion on the possible dynamics in the different regions performed in the previous sections, how to obtain the curves associated with the different cycles as, starting from the cycles of first complexity level, we can obtain all curves of cycles of any complexity level, following the standard adding mechanism. We can therefore state the following

Property 9. *Consider map F_0 with $|1 + S^1| > 1$ and $|1 + S^1 + S^2| < 1$. Then, using the adding scheme, we can write all analytic curves in parameter plane (S^1, S^2) for all periodic orbits of any level of complexity. In any case, for suitable integers p and q, they can all be written as follows:*

$$(1 + S^1)^p (1 + S^1 + S^2)^q = 1.$$

7 Conclusions

In this paper we study a simple financial market model in which interactions between heterogeneous speculators can generate endogenous price dynamics. For two reasons, the model has a discontinuous piecewise linear shape: first, speculators (essentially) rely on linear technical and fundamental trading strategies. Second, while some of them are always active, others stop trading if the misalignment in the market drops below a certain threshold value. One advantage of the model's functional form is that it allows an in-depth and complete analytical investigation of its properties. We find, for instance, that the model cannot produce chaotic motions – although the dynamics appear to be chaotic. Moreover, the model's periodic or quasiperiodic dynamics is structurally unstable, which means that any small change in any parameter of the model leads to a different dynamic behavior. Since our knowledge about discontinuous piecewise linear maps is not yet very deep, we hope that our analysis is also useful for the investigation of similar dynamical systems.

From an economic point of view, we would like to stress that it is quite remarkable that a simple model such as ours can help us to explain the emergence of bubbles and crashes and the high asset price volatility, as observed in many real financial markets. Within our model, bounded endogenous dynamics require that its steady state is unstable. As we have seen, this can be caused by either too aggressive chartists or by too aggressive fundamentalists. Further away from the steady state, the model has, of course, to be stable. Again, this can, in principle, be caused by both the market entry of additional chartists or additional fundamentalists. Additional chartists are beneficial for market stability if the steady state is destabilized by too aggressive fundamentalists while additional fundamentalists are needed for market stability if the steady state is unstable due to the trading behavior of too aggressive chartists.

Our model may be extended in various directions. First, one could assume more complex demand functions and/or more market entry levels. For instance, one could allow chartists to explicitly extrapolate past price trends, which would increase the dimension of our model (the simplest case would be given by a two-dimensional system in which chartists condition their orders on the last observable price change. Alternatively, one could consider different market entry levels for chartists and fundamentalists. As a result, the model would then still be piecewise linear, but instead of having three linear branches, it would have five. Second, one could – and this should clearly be done in the future – try to bring our model (as well as related models) closer to the data. So far, our model is able to mimic certain properties of financial markets, such as bubbles and crashes and excess volatility, in a qualitative manner. An interesting question is thus whether our model, buffeted with some kind of dynamic noise, can also reproduce such salient features in a quantitative way. Finally, it could be interesting to explore our model's policy implications in more detail. According to conventional wisdom, it is essentially the behavior of chartists which destabilizes financial markets; reducing their trading activity is what is required to obtain calmer markets. Yet, as our model shows, the trading activity of chartists can also contribute to market stability. Causalities acting inside financial markets are apparently more complicated than one is tempted to believe, indicating that more research in this direction is needed to improve our understanding of how financial markets function.

References

Boswijk, P., Hommes, C. and Manzan, S., 2007. Behavioral heterogeneity in stock prices. *Journal of Economic Dynamics Control*, 31, 1938-1970.

Brock, W. and Hommes, C., 1997. A rational route to randomness. *Econometrica*, 65, 1059-1095.

Brock, W. and Hommes, C., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics Control*, 22, 1235-1274.

Chiarella, C., 1992. The dynamics of speculative behavior. *Annals of Operations Research*, 37, 101-123.

Chiarella, C., Dieci, R. and He, X.-Z., 2009. Heterogeneity, market mechanisms, and asset price dynamics. In: Hens, T. and Schenk-Hoppe, K.R. (eds): *Handbook of Financial Markets: Dynamics and Evolution.* North-Holland: Amsterdam, 277-344.

Day, R. and Huang, W., 1990. Bulls, bears and market sheep. *Journal of Economic Behavior and Organization*, 14, 299-329.

Day, R., 1997. Complex dynamics, market mediation and stock price behaviour. *North American Actuarial Journal*, 1, 6-21.

De Grauwe, P., Dewachter, H. and Embrechts, M., 1993. *Exchange rate theories. Chaotic models of the foreign exchange market.* Blackwell, Oxford.

Farmer, J. D. and Joshi, S., 2002. The price dynamics of common trading strategies. *Journal of Economic Behavior and Organization*, 49, 149-171.

Gardini, L., Tramontana, F., Avrutin, V. and Schanz, M., 2010. Border Collision Bifurcations in 1D piecewise-linear maps and Leonov's approach. *International Journal of Bifurcation & Chaos* 20, 3085-3104.

He, X.T. and Li, Y., 2008. Heterogeneity, convergence and autocorrelations. *Quantitative Finance*, 8, 59-79.

Hommes, C., 2006. Heterogeneous agent models in economics and finance. In: Tesfatsion, L. and Judd, K. (eds): *Handbook of Computational Economics, Vol. 2: Agent-Based Computational Economics.* North-Holland: Amsterdam, 1107-1186.

Huang, W. and Day, R., 1993. Chaotically switching bear and bull markets: the derivation of stock price distributions from behavioral rules. In: Day, R. and Chen, P. (eds): *Nonlinear dynamics and evolutionary economics*, Oxford University Press, Oxford, 169-182.

Huang, W., Zheng, H. and Chia, W.M., 2010. Financial crisis and interacting heterogeneous agents. *Journal of Economic Dynamics and Control*, 34, 1105-1122.

Keener, J. P., 1980. Chaotic behavior in piecewise continuous difference equations. *Transactions of the American Mathematical Society*, 261(2), 589–604.

Kirman, A., 1993. Ants, rationality, and recruitment. *Quarterly Journal of Economics*, 108, 137-156.

LeBaron, B., Arthur, B. and Palmer, R., 1999. Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, 23, 1487-1516.

LeBaron, B., 2006. Agent-based computational finance. In: Tesfatsion, L. and Judd, K. (eds): *Handbook of Computational Economics, Vol. 2: Agent-Based Computational Economics.* North-Holland: Amsterdam, 1187-1233.

Lux, T., 1995. Herd behaviour, bubbles and crashes. *Economic Journal*, 105, 881-896.

Lux, T., 2009. Stochastic behavioural asset-pricing models and the stylized facts. In: Hens, T. and Schenk-Hoppe, K.R. (eds.): *Handbook of Financial Markets: Dynamics and Evolution.* North-Holland: Amsterdam, 161-216.

Lux, T. and Marchesi, M., 1999. Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, 397, 498-500.

Menkhoff, L. and Taylor, M., 2007. The obstinate passion of foreign exchange professionals: technical analysis. *Journal of Economic Literature*, 45, 936-972.

Nusse, H.E. and Yorke, J.A., 1992. Border-collision bifurcations including period two to period three for piecewise smooth systems, *Physica D*, 57, 39-57.

Nusse, H.E. and Yorke, J.A., 1995. Border-collision bifurcations for piecewise smooth one-dimensional maps, *International Journal of Bifurcation and Chaos*, 5, 189-207.

Sharkovsky A.N., Kolyada, S.F., Sivak, A.G. and Fedorenko, V.V., 1997. *Dynamics of One-dimensional Maps*, Kluwer Academic Publ., Boston.

Shiller, R., 2005. *Irrational Exuberance*, 2nd ed. Princeton University Press, Princeton.

Sushko, I, Gardini, L. [2010] Degenerate Bifurcations and Border Collisions in Piecewise Smooth 1D and 2D Maps", *International Journal of Bifurcation & Chaos* 20(7), 2045–2070.

Tramontana, F., Westerhoff, F. and Gardini, L., 2010. On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. *Journal of Economic Behavior and Organization*, 74, 187-205.

Westerhoff, F., 2009. Exchange rate dynamics: A nonlinear survey. In: Rosser, J.B., Jr. (ed.): *Handbook of Research on Complexity*. Edward Elgar: Cheltenham, 287-325.

Westerhoff, F. and Franke, R., 2011. Converse trading strategies, intrinsic noise and the stylized facts of financial markets. *Quantitative Finance*, in press.