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**Exchangeable Sequences Driven by an Absolutely
Continuous Random Measure**

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EXCHANGEABLE SEQUENCES DRIVEN BY AN ABSOLUTELY CONTINUOUS RANDOM MEASURE

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ABSTRACT. Let S be a Polish space and $(X_n : n \geq 1)$ an exchangeable sequence of S -valued random variables. Let $\alpha_n(\cdot) = P(X_{n+1} \in \cdot \mid X_1, \dots, X_n)$ be the predictive measure and α a random probability measure on S such that $\alpha_n \xrightarrow{weak} \alpha$ a.s.. Two (related) problems are addressed. One is to give conditions for $\alpha \ll \lambda$ a.s., where λ is a (non random) σ -finite Borel measure on S . Such conditions should concern the finite dimensional distributions $\mathcal{L}(X_1, \dots, X_n)$, $n \geq 1$, only. The other problem is to investigate whether $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$, where $\|\cdot\|$ is total variation norm. Various results are obtained. Some of them do not require exchangeability, but hold under the weaker assumption that (X_n) is conditionally identically distributed, in the sense of [2].

1. TWO RELATED PROBLEMS

Throughout, S is a Polish space and

$$X = (X_1, X_2, \dots)$$

a sequence of S -valued random variables on the probability space (Ω, \mathcal{A}, P) . We let \mathcal{B} denote the Borel σ -field on S and \mathbb{S} the set of probability measures on \mathcal{B} . A random probability measure on S is a map $\alpha : \Omega \rightarrow \mathbb{S}$ such that $\sigma(\alpha) \subset \mathcal{A}$, where $\sigma(\alpha)$ is the σ -field on Ω generated by $\omega \mapsto \alpha(\omega)(B)$ for all $B \in \mathcal{B}$.

For each $n \geq 1$, let α_n be the n -th *predictive measure*. Thus, α_n is a random probability measure on S and $\alpha_n(\cdot)(B)$ is a version of $P(X_{n+1} \in B \mid X_1, \dots, X_n)$ for all $B \in \mathcal{B}$. Define also $\alpha_0(\cdot) = P(X_1 \in \cdot)$.

If X is *exchangeable*, as assumed in this section, there is a random probability measure α on S such that

$$\alpha_n(\omega) \xrightarrow{weak} \alpha(\omega) \quad \text{for almost all } \omega \in \Omega.$$

Such an α also grants the usual representation

$$P(X \in B) = \int \alpha(\omega)^\infty(B) P(d\omega) \quad \text{for every Borel set } B \subset S^\infty$$

where $\alpha(\omega)^\infty = \alpha(\omega) \times \alpha(\omega) \times \dots$

Let λ be a σ -finite measure on \mathcal{B} . Our *first problem* is to give conditions for

$$(1) \quad \alpha(\omega) \ll \lambda \quad \text{for almost all } \omega \in \Omega.$$

The conditions should concern the finite dimensional distributions $\mathcal{L}(X_1, \dots, X_n)$, $n \geq 1$, only.

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While investigating (1), one meets another problem, of possible independent interest. Let $\|\cdot\|$ denote total variation norm on (S, \mathcal{B}) . Our *second problem* is to give conditions for

$$\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0.$$

2. MOTIVATIONS

Again, let $X = (X_1, X_2, \dots)$ be exchangeable.

Reasonable conditions for (1) look of theoretical interest. They are of practical interest as well, as regards Bayesian nonparametrics. In this framework, the starting point is a prior π on \mathbb{S} . Since $\pi = P \circ \alpha^{-1}$, condition (1) means that the prior is supported by those $\nu \in \mathbb{S}$ such that $\nu \ll \lambda$. This is a basic information for the subsequent statistical analysis. Roughly speaking, it means that the "underlying statistical model" consists of absolutely continuous laws.

From a foundational point of view, according to de Finetti, only assumptions on observable facts make sense. This is why the conditions for (1) have been requested to concern $\mathcal{L}(X_1, \dots, X_n)$, $n \geq 1$, only. See [3], [5], [6], [7], [8].

A condition of this type is

$$(2) \quad \mathcal{L}(X_1, \dots, X_n) \ll \lambda^n \quad \text{for all } n \geq 1,$$

where $\lambda^n = \lambda \times \dots \times \lambda$. Clearly, (2) is necessary for (1). A (natural) question, thus, is whether (2) suffices for (1) as well.

The answer is yes provided α can be approximated by the predictive measures α_n in some stronger sense. In fact, condition (2) can be written as

$$\alpha_n(\omega) \ll \lambda \quad \text{for all } n \geq 0 \text{ and almost all } \omega \in \Omega.$$

Hence, if (2) holds and $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$, the set

$$A = \{\|\alpha_n - \alpha\| \rightarrow 0\} \cap \{\alpha_n \ll \lambda \text{ for all } n \geq 0\}$$

has probability 1. And, for each $\omega \in A$, one obtains

$$\alpha(\omega)(B) = \lim_n \alpha_n(\omega)(B) = 0 \quad \text{whenever } B \in \mathcal{B} \text{ and } \lambda(B) = 0.$$

Therefore, (1) follows from (2) and $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$. In addition, a martingale argument implies the converse implication, that is

$$\alpha \ll \lambda \text{ a.s.} \iff \|\alpha_n - \alpha\| \xrightarrow{a.s.} 0 \text{ and } \mathcal{L}(X_1, \dots, X_n) \ll \lambda^n \text{ for all } n;$$

see Theorem 1. Thus, our first problem turns into the second one.

The question of whether $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ is of independent interest. Among other things, it is connected to Bayesian consistency. Surprisingly, however, this question seems not answered so far. To the best of our knowledge, $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ in every example known so far. And in fact, for some time, we conjectured that $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ under condition (2). But this is not true. As shown in Example 5, when $S = \mathbb{R}$ and $\lambda =$ Lebesgue measure, it may be that $\mathcal{L}(X_1, \dots, X_n)$ is absolutely continuous for all n and yet α is singular continuous a.s.. Indeed, the (topological) support of $\alpha(\omega)$ has Hausdorff dimension 0 for almost all $\omega \in \Omega$.

Thus, (2) does not suffice for (1). To get (1), in addition to (2), one needs some growth conditions on the conditional densities. We refer to forthcoming Theorem

4 for such conditions. Here, we mention a result on the second problem. Actually, for $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$, it suffices that

$$P\{\omega : \alpha_c(\omega) \ll \lambda\} = 1$$

where $\alpha_c(\omega)$ denotes the continuous part of $\alpha(\omega)$; see Theorem 2.

Finally, some results mentioned above do not need exchangeability of X , but the weaker assumption

$$(X_1, \dots, X_n, X_{n+2}) \sim (X_1, \dots, X_n, X_{n+1}) \quad \text{for all } n \geq 0.$$

Those sequences X satisfying the above condition, investigated in [2], are called *conditionally identically distributed* (c.i.d.).

3. MIXTURES OF I.I.D. ABSOLUTELY CONTINUOUS SEQUENCES

In this section, $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $\mathcal{G}_n = \sigma(X_1, \dots, X_n)$ for $n \geq 1$ and $\mathcal{G}_\infty = \sigma(\cup_n \mathcal{G}_n)$. If μ is a random probability measure on S , we write $\mu(B)$ to denote the real random variable $\mu(\cdot)(B)$, $B \in \mathcal{B}$. Similarly, if $h : S \rightarrow \mathbb{R}$ is a Borel function, integrable with respect to $\mu(\omega)$ for almost all $\omega \in \Omega$, we write $\mu(h)$ to denote $\int h(x) \mu(\cdot)(dx)$.

3.1. Preliminaries. Let $X = (X_1, X_2, \dots)$ be c.i.d., as defined in Section 2. Since X needs not be exchangeable, the representation $P(X \in \cdot) = \int \alpha(\omega)^\infty(\cdot) P(d\omega)$ can fail for any α . However, there is a random probability measure α on S such that

$$(3) \quad \sigma(\alpha) \subset \mathcal{G}_\infty \quad \text{and} \quad \alpha_n(B) = E\{\alpha(B) \mid \mathcal{G}_n\} \quad \text{a.s.}$$

for all $B \in \mathcal{B}$. In particular, $\alpha_n \xrightarrow{weak} \alpha$ a.s.. Also, letting

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

be the empirical measure, one obtains $\mu_n \xrightarrow{weak} \alpha$ a.s.. Such an α is of interest for one more reason. There is an exchangeable sequence $Y = (Y_1, Y_2, \dots)$ of S -valued random variables on (Ω, \mathcal{A}, P) such that

$$(X_n, X_{n+1}, \dots) \xrightarrow{d} Y \quad \text{and} \quad P(Y \in \cdot) = \int \alpha(\omega)^\infty(\cdot) P(d\omega).$$

See [2] for details.

We next recall some known facts about vector-valued martingales; see [9]. Let $(\mathcal{Z}, \|\cdot\|_*)$ be a separable Banach space. Also, let $\mathcal{F} = (\mathcal{F}_n)$ be a filtration and (Z_n) a sequence of \mathcal{Z} -valued random variables on (Ω, \mathcal{A}, P) such that $E\|Z_n\|_* < \infty$ for all n . Then, (Z_n) is an \mathcal{F} -martingale in case $(\phi(Z_n))$ is an \mathcal{F} -martingale for each linear continuous functional $\phi : \mathcal{Z} \rightarrow \mathbb{R}$. If (Z_n) is an \mathcal{F} -martingale, $(\|Z_n\|_*)$ is a real-valued \mathcal{F} -submartingale. So, Doob's maximal inequality yields

$$E\left\{\sup_n \|Z_n\|_*^p\right\} \leq \left(\frac{p}{p-1}\right)^p \sup_n E\{\|Z_n\|_*^p\} \quad \text{for all } p > 1.$$

The following martingale convergence theorem is available as well. Let $Z : \Omega \rightarrow \mathcal{Z}$ be \mathcal{F}_∞ -measurable and such that $E\|Z\|_* < \infty$, where $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. Then, $Z_n \xrightarrow{a.s.} Z$ provided $\phi(Z_n) = E\{\phi(Z) \mid \mathcal{F}_n\}$ a.s. for all n and all linear continuous functionals $\phi : \mathcal{Z} \rightarrow \mathbb{R}$.

3.2. Results. In the sequel, λ is a σ -finite measure on \mathcal{B} and α a random probability measure on S such that $\alpha_n \xrightarrow{weak} \alpha$ a.s.. Equivalently, if X is c.i.d. (in particular, exchangeable), α is a random probability measure on S such that $\mu_n \xrightarrow{weak} \alpha$ a.s.. It can (and will) be assumed $\sigma(\alpha) \subset \mathcal{G}_\infty$.

Theorem 1. *Suppose $X = (X_1, X_2, \dots)$ is c.i.d.. Then, $\alpha \ll \lambda$ a.s. if and only if $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ and $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$ for all n .*

Proof. The "if" part can be proved exactly as in Section 2. Conversely, suppose $\alpha \ll \lambda$ a.s.. It can be assumed $\alpha(\omega) \ll \lambda$ for all $\omega \in \Omega$. We let $L_p = L_p(S, \mathcal{B}, \lambda)$ for each $1 \leq p \leq \infty$.

Let $f : \Omega \times S \rightarrow [0, \infty)$ be such that $\alpha(\omega)(dx) = f(\omega, x) \lambda(dx)$ for all $\omega \in \Omega$. Since \mathcal{B} is countably generated, f can be taken $\mathcal{A} \otimes \mathcal{B}$ -measurable (see [4], V.5.58, page 52) so that

$$1 = \int 1 dP = \int \int f(\omega, x) \lambda(dx) P(d\omega) = \int E\{f(\cdot, x)\} \lambda(dx).$$

Thus, given $n \geq 0$, $E\{f(\cdot, x) \mid \mathcal{G}_n\}$ is well defined for λ -almost all $x \in S$. Since X is c.i.d., condition (3) also implies

$$\begin{aligned} \int_B E\{f(\cdot, x) \mid \mathcal{G}_n\} \lambda(dx) &= E\left\{\int_B f(\cdot, x) \lambda(dx) \mid \mathcal{G}_n\right\} \\ &= E\{\alpha(B) \mid \mathcal{G}_n\} = \alpha_n(B) \quad \text{a.s. for fixed } B \in \mathcal{B}. \end{aligned}$$

Since \mathcal{B} is countably generated, the previous equality yields

$$\alpha_n(\omega)(dx) = E\{f(\cdot, x) \mid \mathcal{G}_n\}(\omega) \lambda(dx) \quad \text{for almost all } \omega \in \Omega.$$

This proves that $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$ for all n . In particular, up to modifying α_n on a P -null set, it can be assumed $\alpha_n(\omega)(dx) = f_n(\omega, x) \lambda(dx)$ for all $n \geq 0$, all $\omega \in \Omega$, and suitable functions $f_n : \Omega \times S \rightarrow [0, \infty)$.

Regard $f, f_n : \Omega \rightarrow L_1$ as L_1 -valued random variables. Then, $f : \Omega \rightarrow L_1$ is \mathcal{G}_∞ -measurable for $\int h(x) f(\cdot, x) \lambda(dx) = \alpha(h)$ is \mathcal{G}_∞ -measurable for all $h \in L_\infty$. Clearly, $\|f(\omega, \cdot)\|_{L_1} = \|f_n(\omega, \cdot)\|_{L_1} = 1$ for all n and ω . Finally, X c.i.d. implies

$$\begin{aligned} E\left\{\int h(x) f(\cdot, x) \lambda(dx) \mid \mathcal{G}_n\right\} &= E\{\alpha(h) \mid \mathcal{G}_n\} = \alpha_n(h) \\ &= \int h(x) f_n(\cdot, x) \lambda(dx) \quad \text{a.s. for all } h \in L_\infty. \end{aligned}$$

By the martingale convergence theorem (see Subsection 3.1) $f_n \xrightarrow{a.s.} f$ in the space L_1 , that is

$$\|\alpha_n(\omega) - \alpha(\omega)\| = \frac{1}{2} \int |f_n(\omega, x) - f(\omega, x)| \lambda(dx) \longrightarrow 0 \quad \text{for almost all } \omega \in \Omega.$$

□

In the exchangeable case, the argument of the previous proof yields a little bit more. Indeed, if X is exchangeable and $\alpha \ll \lambda$ a.s., then

$$\sup_{B \in \mathcal{B}^k} \left| P\{(X_{n+1}, \dots, X_{n+k}) \in B \mid \mathcal{G}_n\} - \alpha^k(B) \right| \xrightarrow{a.s.} 0,$$

where $k \geq 1$ is any integer and $\alpha^k = \alpha \times \dots \times \alpha$.

Next result deals with the second problem of Section 1. For each $\nu \in \mathbb{S}$, let ν_c and ν_d denote the continuous and discrete parts of ν , that is, $\nu_d(B) = \sum_{x \in B} \nu\{x\}$ for all $B \in \mathcal{B}$ and $\nu_c = \nu - \nu_d$.

Theorem 2. *Suppose $X = (X_1, X_2, \dots)$ is c.i.d. and $P\{\omega : \alpha_c(\omega) \ll \lambda\} = 1$. Then, $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ if and only if*

$$(4) \quad \begin{aligned} & \text{there is a set } A_0 \in \mathcal{A} \text{ such that } P(A_0) = 1 \text{ and} \\ & \alpha_n(\omega)\{x\} \longrightarrow \alpha(\omega)\{x\} \text{ for all } x \in S \text{ and } \omega \in A_0. \end{aligned}$$

In particular, $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ if X is exchangeable and $\alpha_c \ll \lambda$ a.s. (in fact, condition (4) is automatically true if X is exchangeable).

Proof. The "only if" part is trivial. Suppose condition (4) holds. For each $n \geq 0$, take functions β_n and γ_n on Ω such that $\beta_n(\omega)$ and $\gamma_n(\omega)$ are measures on \mathcal{B} for all $\omega \in \Omega$ and

$$\beta_n(B) = E\{\alpha_c(B) \mid \mathcal{G}_n\}, \quad \gamma_n(B) = E\{\alpha_d(B) \mid \mathcal{G}_n\}, \quad \text{a.s.},$$

for all $B \in \mathcal{B}$. Since X is c.i.d., condition (3) yields $\alpha_n = \beta_n + \gamma_n$ a.s..

We first prove $\|\beta_n - \alpha_c\| \xrightarrow{a.s.} 0$. It can be assumed $\alpha_c(\omega) \ll \lambda$ for all $\omega \in \Omega$, so that $\alpha_c(\omega)(dx) = f(\omega, x) \lambda(dx)$ for all $\omega \in \Omega$ and some function $f : \Omega \times S \rightarrow [0, \infty)$. For fixed $B \in \mathcal{B}$, arguing as in the proof of Theorem 1, one has

$$\beta_n(B) = E\left\{\int_B f(\cdot, x) \lambda(dx) \mid \mathcal{G}_n\right\} = \int_B E(f(\cdot, x) \mid \mathcal{G}_n) \lambda(dx) \quad \text{a.s..}$$

By standard arguments, it follows that $\beta_n \ll \lambda$ a.s.. Again, it can be assumed $\beta_n(\omega)(dx) = f_n(\omega, x) \lambda(dx)$ for all $\omega \in \Omega$ and some function $f_n : \Omega \times S \rightarrow [0, \infty)$. Define $L_1 = L_1(S, \mathcal{B}, \lambda)$ and regard $f_n, f : \Omega \rightarrow L_1$ as L_1 -valued random variables. By the same martingale argument used for Theorem 1, one obtains $f_n \xrightarrow{a.s.} f$ in the space L_1 . That is, $\|\beta_n - \alpha_c\| \xrightarrow{a.s.} 0$.

We next prove $\|\gamma_n - \alpha_d\| \xrightarrow{a.s.} 0$. Take A_0 as in condition (4) and define

$$A_1 = \left\{ \lim_n \|f_n - f\|_{L_1} = 0 \text{ and } \alpha_n = \beta_n + \gamma_n \text{ for all } n \geq 0 \right\}.$$

Then, $P(A_0 \cap A_1) = 1$ and

$$\begin{aligned} \alpha_d(\omega)\{x\} &= \alpha(\omega)\{x\} - \alpha_c(\omega)\{x\} = \alpha(\omega)\{x\} - f(\omega, x) \lambda\{x\} \\ &= \lim_n (\alpha_n(\omega)\{x\} - f_n(\omega, x) \lambda\{x\}) = \lim_n (\alpha_n(\omega)\{x\} - \beta_n(\omega)\{x\}) = \lim_n \gamma_n(\omega)\{x\} \end{aligned}$$

for all $\omega \in A_0 \cap A_1$ and $x \in S$. Define also

$$A = A_0 \cap A_1 \cap \{\gamma_n(S) \longrightarrow \alpha_d(S)\}.$$

Since $\gamma_n(S) = 1 - \beta_n(S) \xrightarrow{a.s.} 1 - \alpha_c(S) = \alpha_d(S)$, then $P(A) = 1$. Fix $\omega \in A$ and let $D_\omega = \{x \in S : \alpha(\omega)\{x\} > 0\}$. Then,

$$\alpha_d(\omega)(D_\omega) \leq \liminf_n \gamma_n(\omega)(D_\omega)$$

since D_ω is countable and $\alpha_d(\omega)\{x\} = \lim_n \gamma_n(\omega)\{x\}$ for all $x \in D_\omega$. Further,

$$\limsup_n \gamma_n(\omega)(D_\omega) \leq \limsup_n \gamma_n(\omega)(S) = \alpha_d(\omega)(S) = \alpha_d(\omega)(D_\omega).$$

Therefore, $\lim_n \|\gamma_n(\omega) - \alpha_d(\omega)\| = 0$ is an immediate consequence of

$$\begin{aligned} \gamma_n(\omega)\{x\} &\longrightarrow \alpha_d(\omega)\{x\} \quad \text{for each } x \in D_\omega, \\ \alpha_d(\omega)(D_\omega) &= \lim_n \gamma_n(\omega)(D_\omega), \quad \alpha_d(\omega)(D_\omega^c) = \lim_n \gamma_n(\omega)(D_\omega^c) = 0. \end{aligned}$$

Finally, suppose X exchangeable. We have to prove condition (4). If S is countable, condition (4) is trivial for $\alpha_n(B) \xrightarrow{a.s.} \alpha(B)$ for fixed $B \in \mathcal{B}$. If $S = \mathbb{R}$, Glivenko-Cantelli theorem yields $\sup_x |\mu_n(I_x) - \alpha(I_x)| \xrightarrow{a.s.} 0$, where $I_x = (-\infty, x]$ and $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ is the empirical measure. Hence, (4) follows from

$$\sup_x |\alpha_n(I_x) - \mu_n(I_x)| \xrightarrow{a.s.} 0;$$

see Corollary 3.2 of [1]. If S is any uncountable Polish space, take a Borel isomorphism $\psi : S \rightarrow \mathbb{R}$. (Thus, ψ is bijective with ψ and ψ^{-1} Borel measurable). Then, $(\psi(X_n))$ is an exchangeable sequence of real random variables and condition (4) is a straightforward consequence of

$$P\{\psi(X_{n+1}) \in B \mid \psi(X_1), \dots, \psi(X_n)\} = P\{\psi(X_{n+1}) \in B \mid \mathcal{G}_n\} = \alpha_n(\psi^{-1}B) \quad \text{a.s.}$$

for each Borel set $B \subset \mathbb{R}$. This concludes the proof. \square

When X is c.i.d. (but not exchangeable) $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ needs not be true even if $\alpha_c \ll \lambda$ a.s..

Example 3. Let (Z_n) and (U_n) be independent sequences of independent real random variables such that $Z_n \sim \mathcal{N}(0, b_n - b_{n-1})$ and $U_n \sim \mathcal{N}(0, 1 - b_n)$, where $0 = b_0 < b_1 < b_2 < \dots < 1$ and $\sum_n (1 - b_n) < \infty$. As shown in Example 1.2 of [2],

$$X_n = \sum_{i=1}^n Z_i + U_n$$

is c.i.d. and $X_n \xrightarrow{a.s.} V$ for some real random variable V . Since $\mu_n \xrightarrow{weak} \delta_V$ a.s., then $\alpha = \delta_V$ and $\alpha_c \ll \lambda$ a.s. (in fact, $\alpha_c = 0$ a.s.). However, condition (4) fails. In fact, $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$ for all n , where λ is Lebesgue measure. Hence, $\alpha_n(\omega)\{V(\omega)\} = 0$ while $\alpha(\omega)\{V(\omega)\} = 1$ for all n and almost all $\omega \in \Omega$.

We now turn to the first problem of Section 1. Recall that condition (2) amounts to $\alpha_n \ll \lambda$ a.s. for all $n \geq 0$. Therefore, up to modifying α_n on a P -null set, under condition (2) one can write

$$\alpha_n(\omega)(dx) = f_n(\omega, x) \lambda(dx)$$

for each $\omega \in \Omega$, each $n \geq 0$, and some function $f_n : \Omega \times S \rightarrow [0, \infty)$.

Theorem 4. Suppose $X = (X_1, X_2, \dots)$ is c.i.d. and $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$ for all n . Fix a constant $p > 1$ and define

$$I_n^B(\omega) = \int_B f_n(\omega, x)^p \lambda(dx), \quad B \in \mathcal{B}.$$

Then, $\alpha \ll \lambda$ a.s. provided, for every compact $K \subset S$,

$$(5) \quad \sup_n I_n^K(\omega) < \infty \quad \text{for almost all } \omega \in \Omega.$$

In particular, $\alpha \ll \lambda$ a.s. whenever $\sup_n E\{I_n^K\} < \infty$ for each compact $K \subset S$.

Proof. Fix a nondecreasing sequence $B_1 \subset B_2 \subset \dots$ such that $B_n \in \mathcal{B}$, $\lambda(B_n) < \infty$, and $\cup_n B_n = S$. Since $\lambda(B_1) < \infty$ and S is Polish, there is a compact set $K_1 \subset B_1$ satisfying $\lambda(B_1 \cap K_1^c) < 1$. By induction, for each $n \geq 2$, there is a compact set K_n such that $K_{n-1} \subset K_n \subset B_n$ and $\lambda(B_n \cap K_n^c) < 1/n$. Since X is c.i.d., condition (3) implies

$$\alpha(K_m) = \lim_n E\{\alpha(K_m) \mid \mathcal{G}_n\} = \lim_n \alpha_n(K_m) \quad \text{a.s. for all } m \geq 1.$$

Define $H = \cup_m K_m$ and $A_H = \{\alpha(H) = 1\}$. If $\omega \in A_H$, then

$$\alpha(\omega)(B) = \alpha(\omega)(B \cap H) = \sup_m \alpha(\omega)(B \cap K_m) \quad \text{for all } B \in \mathcal{B}.$$

Moreover, $P(A_H) = 1$. In fact, $\lambda(H^c) = 0$ and $\alpha_n \ll \lambda$ a.s. for all n , so that

$$\alpha(H) = \lim_n E\{\alpha(H) \mid \mathcal{G}_n\} = \lim_n \alpha_n(H) = 1 \quad \text{a.s.}$$

Thus, it suffices to prove $\alpha(\cdot \cap K_m) \ll \lambda$ a.s. for all m .

Suppose (5) holds. Fix $m \geq 1$ and define $K = K_m$ and $\lambda_K(\cdot) = \lambda(\cdot \cap K)$. By (5) and $p > 1$, the sequence $(f_n(\omega, \cdot) : n \geq 1)$ is uniformly integrable in $(S, \mathcal{B}, \lambda_K)$ for almost all $\omega \in \Omega$. Take a set $A \in \mathcal{A}$ such that $P(A) = 1$ and, for each $\omega \in A$,

$$\begin{aligned} \alpha(\omega)(K) &= \lim_n \alpha_n(\omega)(K), \quad \alpha_n(\omega) \xrightarrow{weak} \alpha(\omega), \\ (f_n(\omega, \cdot) : n \geq 1) &\text{ is uniformly integrable in } (S, \mathcal{B}, \lambda_K). \end{aligned}$$

Fix $\omega \in A$. Since $\lambda_K(S) = \lambda(K) \leq \lambda(B_m) < \infty$ and $(f_n(\omega, \cdot) : n \geq 1)$ is uniformly integrable, there is a subsequence (n_j) and a function $\psi_\omega \in L_1(S, \mathcal{B}, \lambda_K)$ such that $f_{n_j}(\omega, \cdot) \rightarrow \psi_\omega$ in the weak-topology of $L_1(S, \mathcal{B}, \lambda_K)$. This means that

$$\int_{B \cap K} \psi_\omega(x) \lambda(dx) = \lim_j \int_{B \cap K} f_{n_j}(\omega, x) \lambda(dx) = \lim_j \alpha_{n_j}(\omega)(B \cap K) \quad \text{for all } B \in \mathcal{B}.$$

Therefore,

$$\begin{aligned} \int_K \psi_\omega(x) \lambda(dx) &= \lim_j \alpha_{n_j}(\omega)(K) = \alpha(\omega)(K) \quad \text{and} \\ \int_{F \cap K} \psi_\omega(x) \lambda(dx) &= \lim_j \alpha_{n_j}(\omega)(F \cap K) \leq \alpha(\omega)(F \cap K) \quad \text{for each closed } F \subset S. \end{aligned}$$

By standard arguments, the previous two relations yield $\alpha(\omega)(B \cap K) = \int_{B \cap K} \psi_\omega(x) \lambda(dx)$ for all $B \in \mathcal{B}$. Thus, $\alpha(\omega)(\cdot \cap K) \ll \lambda$. This concludes the proof of the first part.

It remains to see that condition (5) follows from $\sup_n E\{I_n^K\} < \infty$ for each compact K . Fix $B \in \mathcal{B}$ and suppose $\sup_n E\{I_n^B\} < \infty$. Let $\lambda_B(\cdot) = \lambda(\cdot \cap B)$ and $L_r = L_r(S, \mathcal{B}, \lambda_B)$ for all r . It can be assumed $I_n^B(\omega) < \infty$ for all $\omega \in \Omega$ and $n \geq 0$. Thus, each $f_n : \Omega \rightarrow L_p$ can be seen as an L_p -valued random variable such that

$$E\|f_n\|_{L_p} = E\{(I_n^B)^{1/p}\} \leq (E\{I_n^B\})^{1/p} < \infty.$$

Further, $\int f_n(\cdot, x) h(x) \lambda_B(dx) = \alpha_n(I_B h)$ is \mathcal{G}_n -measurable for all $h \in L_q$, where $q = p/(p-1)$. Since X is c.i.d., condition (3) also implies

$$\begin{aligned} E\left\{\int f_{n+1}(\cdot, x) h(x) \lambda_B(dx) \mid \mathcal{G}_n\right\} &= E\{\alpha_{n+1}(I_B h) \mid \mathcal{G}_n\} \\ &= E\{E(\alpha(I_B h) \mid \mathcal{G}_{n+1}) \mid \mathcal{G}_n\} \\ &= E\{\alpha(I_B h) \mid \mathcal{G}_n\} = \alpha_n(I_B h) \\ &= \int f_n(\cdot, x) h(x) \lambda_B(dx) \quad \text{a.s. for all } h \in L_q. \end{aligned}$$

Hence, (f_n) is a (\mathcal{G}_n) -martingale. By Doob's maximal inequality,

$$E\{\sup_n I_n^B\} = E\{\sup_n \|f_n\|_{L_p}^p\} \leq q^p \sup_n E\{\|f_n\|_{L_p}^p\} = q^p \sup_n E\{I_n^B\} < \infty.$$

In particular, $\sup_n I_n^B < \infty$ a.s., and this concludes the proof. \square

Some remarks on Theorem 4 are in order. First,

$$f_n(\omega, \cdot) = \frac{g_{n+1}(X_1(\omega), \dots, X_n(\omega), \cdot)}{g_n(X_1(\omega), \dots, X_n(\omega))} \quad \text{for almost all } \omega \in \Omega,$$

where each $g_n : S^n \rightarrow [0, \infty)$ is a density of $\mathcal{L}(X_1, \dots, X_n)$ with respect to λ^n . Thus, more concretely, I_n^B can be written as

$$I_n^B = \frac{\int_B g_{n+1}(X_1, \dots, X_n, x)^p \lambda(dx)}{g_n(X_1, \dots, X_n)^p} \quad \text{a.s..}$$

Second, as apparent from the proof, condition (5) can be slightly weakened as follows. For each compact K , the sequence $(f_n(\omega, \cdot) : n \geq 1)$ is uniformly integrable, in the space $(S, \mathcal{B}, \lambda_K)$, for almost all $\omega \in \Omega$.

Third, suppose X exchangeable and fix *any* random probability measure γ on S such that $P(X \in \cdot) = \int \gamma(\omega)^\infty(\cdot) P(d\omega)$. Then, $\gamma \ll \lambda$ a.s. under the assumptions of Theorem 4. In fact, α and γ have the same probability distribution, when regarded as \mathbb{S} -valued random variables.

A last (and important) remark deals with condition (2). Indeed, even if X is exchangeable, condition (2) is not enough for $\alpha \ll \lambda$ a.s.. When $S = \mathbb{R}$ and $\lambda =$ Lebesgue measure, it may be that X is exchangeable, $\mathcal{L}(X_1, \dots, X_n)$ is absolutely continuous for all n , and yet the support of $\alpha(\omega)$ has Hausdorff dimension 0 for almost all $\omega \in \Omega$. We close the paper showing this fact.

Example 5. Let U_m and $Y_{m,n}$ be independent real random variables, on the probability space (Ω, \mathcal{A}, P) , such that:

- U_m is uniformly distributed on $(\frac{1}{m+1}, \frac{1}{m})$ for each $m \geq 1$;
- $P(Y_{m,n} = 0) = P(Y_{m,n} = 1) = \frac{1}{2}$ for all $m, n \geq 1$.

Define $V_m = U_m^m$ and

$$X_n = \sum_{m=1}^{\infty} U_m^m Y_{m,n} = \sum_{m=1}^{\infty} V_m Y_{m,n}.$$

Then, $X = (X_1, X_2, \dots)$ is conditionally i.i.d. given $\mathcal{V} = \sigma(V_1, V_2, \dots)$. Precisely, for $\omega \in \Omega$ and $B \in \mathcal{B}$, define

$$\alpha(\omega)(B) = P\left\{u \in \Omega : \sum_m V_m(\omega) Y_{m,1}(u) \in B\right\}.$$

Then, $\alpha(B)$ is a version of $P(X_1 \in B \mid \mathcal{V})$ and $P(X \in \cdot) = \int \alpha(\omega)^\infty(\cdot) P(d\omega)$. In particular, X is exchangeable. Moreover, $\mu_n \xrightarrow{weak} \alpha$ a.s. for

$$P(\mu_n \xrightarrow{weak} \alpha \mid \mathcal{V}) = 1 \quad \text{a.s.}$$

Next, the (topological) support of $\alpha(\omega)$ has Hausdorff dimension 0 for almost all $\omega \in \Omega$. Suppose in fact $b_1 > b_2 > \dots > 0$ are real numbers such that $\sum_m b_m < \infty$ and Z_1, Z_2, \dots i.i.d. random variables with $P(Z_1 = 0) = P(Z_1 = 1) = 1/2$. Then, by Theorem 8 of [10], the support of $\mathcal{L}(\sum_m b_m Z_m)$ has Hausdorff dimension 0 whenever $\lim_m (\sum_{j>m} b_j)^{-1} b_m = \infty$. Thus, letting $b_m = V_m(\omega)$ and $Z_m = Y_{m,1}$, it suffices to verify that

$$(6) \quad \lim_m \frac{V_m(\omega)}{\sum_{j>m} V_j(\omega)} = \infty \quad \text{for almost all } \omega \in \Omega.$$

And condition (6) follows immediately from

$$(j+1)^{-j} < V_j < j^{-j} \quad \text{and} \quad \sum_{j>m} V_j \leq \sum_{j>m} j^{-j} \leq \sum_{j>m} (m+1)^{-j} = \frac{(m+1)^{-m}}{m} \quad \text{a.s.}$$

We finally prove that $\mathcal{L}(X_1, \dots, X_n)$ is absolutely continuous, with respect to Lebesgue measure on \mathbb{R}^n , for all n . Given the array $y = (y_{m,n} : m, n \geq 1)$, with $y_{m,n} \in \{0, 1\}$ for all m, n , define

$$X_{n,y} = \sum_m V_m y_{m,n}.$$

Fix $n \geq 1$ and denote I_n the $n \times n$ identity matrix. If y satisfies

$$(7) \quad \begin{pmatrix} y_{m+1,1} & \cdots & y_{m+1,n} \\ \cdots & \cdots & \cdots \\ y_{m+n,1} & \cdots & y_{m+n,n} \end{pmatrix} = I_n \quad \text{for some } m \geq 0,$$

then

$$(X_{1,y}, \dots, X_{n,y}) = (V_{m+1}, \dots, V_{m+n}) + (R_1, \dots, R_n) \\ \text{with } (R_1, \dots, R_n) \text{ independent of } (V_{m+1}, \dots, V_{m+n}).$$

In this case, since $(V_{m+1}, \dots, V_{m+n})$ has an absolutely continuous distribution, $(X_{1,y}, \dots, X_{n,y})$ has an absolutely continuous distribution as well. Hence, letting $Y = (Y_{m,n} : m, n \geq 1)$, the conditional distribution of (X_1, \dots, X_n) given $Y = y$ is absolutely continuous as far as y satisfies (7). To conclude the proof, it suffices noting that

$$P(Y = y \text{ for some } y \text{ satisfying (7)}) = 1.$$

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