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Some Mathematical Issues on Linear Joint Production Models

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Abstract. We give a survey of some mathematical issues concerning linear joint production models, usually described by a pair of nonnegative matrices (A, B), not necessarily square. The celebrated von Neumann growth model is not included in the present paper. In the last section we review some approaches to the generalization of the Perron-Frobenius problem for a pair (A, B), i. e. the problem $Ax = \lambda Bx$.

Key words: Linear joint production models, generalized Perron-Frobenius theorem.

1. Introduction

Besides the classical von Neumann growth model, which entails joint production, several linear joint production models have been considered in the economic literature, especially with reference to Sraffa's models, examined by P. Sraffa (1060) in the second part of his famous book. In this paper we analyze linear joint production models, described by equalities and involving a pair of nonnegative matrices (A, B), trying to generalize, from a mathematical point of view, what holds for single production models, of the type of Leontief models and Sraffa models with single production (first part of the book of Sraffa). We follow mainly the papers of Bidard (2004), Peris (1991) and Peris and Villar (1993). In Section 2 we give some basic definitions and formal properties of a linear economic model with joint production. Section 3 is concerned with square joint production systems. In Section 4 we examine some extensions to non square systems, i. e. the case in which the number of sectors may be different from the number of commodities (as in the classical von Neumann growth model) and in the final Section 5 we give some hints on Perron-Frobenius results for a pair of matrices. The von Neumann growth model is not considered in the present paper. This subject has been treated in Giorgi and Bidard (2025). We point out that in this paper there is a misprint in Remark 2, at page 9. Instead of "the value of this problem is negative", read "the value of this problem is nonnegative".

The convention for vector comparison is the following one (x and y being two vectors in $\mathbb{R}^n$ and $[0]$ being the zero vector):

a) $x \geqq y$ means $x_i \geqq y_i$, $i = 1, ..., n$. If $x \geqq [0]$ we speak of "nonnegative vectors".

b) $x \geq y$ means $x \geqq y$ and $x_i > y_i$ for some i, i. e. $x \geqq y$ but $x \neq y$. If $x \geq [0]$ we speak of "semipositive vectors".

c) $x > y$ means $x_i > y_i$, $i = 1, ..., n$. If $x > [0]$ we speak of "positive vectors".

The same convention holds for the comparison between two matrices A and B of the same order. A_i, $i = 1, ..., m$, denotes the i-th row of the matrix A of order (m, n), whereas A^j, $j = 1, ..., n$, denotes the j-th column.

2. An economic model of linear joint production

Consider an economic system in which n different production processes are available to produce n commodities. All processes operate under constant return to scale. The economic system in exam is described by two nonnegative matrices A and B, both of order (m, n) :

$$A \geqq [0]\,, \quad B \geqq [0]\,.$$

In these matrices a_{ij} denotes the quantity of the i-th good which enters as a mean of production in the j-th industry, $i = 1, ..., m$; $j = 1, ..., n$, where the j-th production process is activated at a unitary intensity level. The element b_{ij}, $i = 1, ..., m$; $j = 1, ..., n$, denotes the quantity of the i-th good produced by the j-th process, always activated at a unitary intensity level. Hence every column of A, A^j, $j = 1, ..., n$, represents the inputs used by the various processes, whereas every column of B, B^j, $j = 1, ..., n$, represents the outputs obtained by the production processes: therefore the pair (A^j, B^j) represents a production process (activated at a unitary intensity level). The rows of A and B contain the goods, respectively considered as means of production and as products. If $x \in \mathbb{R}^n_+$ represents the n-vector of activity levels, then Bx represents the gross output vector and Ax represents the vector of input requirements. The matrix A may be called the "matrix of inputs" and B may be called the "matrix of outputs". Following Kemeny, Morgenstern and Thompson (1950), it is convenient to specify better the assumption $A \geqq [0]$, $B \geqq [0]$, i. e. to assume

$$A^j \geq [0]\,, \; j = 1, ..., n; \; B_i \geq [0]\,, \; i = 1, ..., m.$$

The economic meaning of the above assumptions is clear: each process uses a good at least and each good is produced in a process at least.

When $m = n$ the above economic system is *square,* i. e. we have a number of sectors equal to the number of commodities, as in the models described by Sraffa (1960) in Chapter 5 of his famous book. The equilibrium system for quantities can be written as

$$Bx = Ax + d \tag{1}$$

where on A and B the above assumptions are made and where $d \in \mathbb{R}^m_+$ is a given final demand (or consumption) vector.

Sraffa (1960) considers also an equilibrium system for prices, i. e. the system

$$p^\top B = (1+r)p^\top A + w\ell^\top ,$$

where $r \geqq 0$ is the *rate of profit* (common to all industries), $w \geqq 0$ is the *wage rate* (common to all industries) and $\ell > [0]$ is the vector of the total labour requirements (as previously said, in Sraffa (1960) the models are square).

Definition 1. The above joint production model is said to be *productive* or that the *productivity condition* holds for the same, if there exists an activity level $x \geq [0]$ (or equivalently $x > [0]$) such that

$$(Bx > Ax) \iff (B-A)x > [0] . \tag{2}$$

This condition says that the system allows to obtain a positive net output vector for some nonnegative gross output vector. An economic system (A, B) which satisfies (2) is called by Bidard and Erreygers (1998a) "strictly viable".

Definition 2. The above joint production model is said to be *profitable* if there exists a vector $p \geq [0]$ such that

$$(p^\top B > p^\top A) \iff p^\top (B-A) > [0] .$$

It must be noted that the model (A, B) is productive if and only if the matrix $(B-A)$ is an $\mathcal{S}$-*matrix* or belongs to the class of $\mathcal{S}$-matrices (following the terminology of Fiedler and Ptàk (1966)) and that the model (A, B) is profitable if and only if the matrix $(B-A)^\top$ belongs to the above class of $\mathcal{S}$-matrices, or if and only if $(B-A)$ belongs to the class of $\mathcal{S}^\top$-matrices. We have to note that in general the classes $\mathcal{S}$ and $\mathcal{S}^\top$ *do not coincide,* but are two independent classes of matrices. If $B = I$, then the two classes of matrices coincide, as it is well known from the anlaysis of linear economic models with single production. For instance, with

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix},$$

we see that the pair (A, B) is profitable but not productive, being

$$(B-A) = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}.$$

On the other hand, if we consider the transpose of A and B, i. e.

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix},$$

we see that the pair (A, B) is productive, but not profitable.

In Section 4 a characterization of productivity and profitability for non necessarily square systems (A, B) will be given. Now we give a sufficient condition for the productivity of the pair (A, B), A and B *square* of order n. We recall that (A, B) is a productive system if and only if $M = (B - A)$ is an S-matrix, in the terminology of Fiedler and Ptàk (1966), i. e. the system

$$\begin{cases} Mx > [0] \\ x \geq [0] \ \ (\text{equivalently: } x > [0]\,) \end{cases}$$

admits a solution. Always following the terminology of Fiedler and Ptàk (1962, 1966) and of Gale and Nikaido (1965), a square matrix M belongs to the $\mathcal{P}$*-class* or is a $\mathcal{P}$*-matrix* if and only if its principal minors are positive, i. e. $-M$ is an *Hicksian matrix* (see, e. g., Giorgi and Zuccotti (2014), Kemp and Kimura (1978), Quirk (1981)). A nice result of Fiedler and Ptàk (1966, Theorem 2.6) is that if $M \in \mathcal{P}$, then $M \in \mathcal{S}$; hence, in our terminology:

$$M = (B - A) \in \mathcal{P} \Longrightarrow M = (B - A) \text{ productive.}$$

Similarly, following Gale (1960), if we define *quasi-productive* or *viable* the pair (A, B) when the system

$$\begin{cases} (B - A)x \geqq [0] \\ x \geq [0] \end{cases}$$

has a solution (in the terminology of Fiedler and Ptàk (1966), the matrix $(B-A)$ belongs to the class $\mathcal{S}_0$; in economic terms the system (A, B) is viable if it is able to reproduce itself) and define, always following Fiedler and Ptàk (1966), the class $\mathcal{P}_0$ as the class of those *square* matrices M which have all principal minors nonnegative, then it can be proved that if $(B - A) \in \mathcal{P}_0$, then $(B - A) \in \mathcal{S}_0$, i. e. $(B - A)$ is viable. In the economic terminology $M \in \mathcal{P}_0$ if and only if $-M$ is *quasi-Hicksian.* The proof of the above implications runs as follows. One of the charatcterizations of the matrices of class $\mathcal{P}$ is the following one (see Berman and Plemmons (1994), Fiedler (1986)).

• For every diagonal matrix S whose diagonal entries are ± 1 ("signature matrix"), there exists a vector $x \geq [0]$ such that $SMSx > [0]$.

It is sufficient to take $S = I$ to obtain the above implication. Similarly for the case of $\mathcal{P}_0$ matrices (see Berman and Plemmons (1994)). The class $\mathcal{P}$ finds important applications also in the so-called *linear complementarity problems.* Given a real square matrix M and a real vector q, find the vectors

$$w \geqq [0]\,, \quad z \geqq [0]$$

such that

$$\begin{aligned} w &= Mz + q \\ w^\top z &= 0. \end{aligned}$$

The solution of the basic problem of linear programming, which consists in minimizing $c^\top x$ over the feasible set $Ax \geqq b, \ \ x \geqq [0]$, can be obtained as a solution of the above linear complementarity problem, with

$$M = \begin{bmatrix} [0] & A \\ -A^\top & [0] \end{bmatrix}; \ \ q = \begin{bmatrix} -b \\ c \end{bmatrix}.$$

3. Square joint production systems

We begin by recalling a result of Plemmons (1977) and Berman and Plemmons (1979) who give several equivalent conditions so that a square matrix M has a semipositive inverse.

Theorem 1. Let be given a square matrix of order n. Then, the following conditions are equivalent.

(1) M is nonsingular and $M^{-1} \geq [0]$.

(2) For each $d \geq [0]$ there exists $x \geq [0]$ such that $Mx = d$.

(3) For each $x \in \mathbb{R}^n$ such that $Mx \geqq [0]$ it holds $x \geqq [0]$. Equivalently: $(Mx - My) \geqq [0] \Longrightarrow x \geqq y$.

(4) There exist two nonsingular matrices E and F, with $E^{-1} \geq [0]$ and $F^{-1} \geq [0]$, such that

$$E \leqq M \leqq F.$$

(5) M admits a *convergent regular splitting,* that is

$$M = B - A, \ \ B^{-1} \geq [0] \, , \ \ A \geq [0] \, ,$$

with AB^{-1} *convergent.* (A square matrix A is convergent if $\lim_{n \longrightarrow +\infty} (A)^n = [0]$. Equivalently, if its spectral radius is less than 1).

(6) M admits a *convergent weak regular splitting,* that is

$$M = B - A, \ \ B^{-1} \geq [0] \, , \ \ AB^{-1} \geq [0] \, ,$$

with AB^{-1} *convergent.*

(7) There exists a nonsingular matrix E, with $E^{-1} \geq [0]$ such that $I - E^{-1}M \geqq [0]$ and this matrix is convergent.

Note that in the decompositions (5) and (6) of Theorem 1 it is not required that A and B are both semipositive. Characterization (3) is known under the name that M is a *monotone matrix* (Collatz (1952, 1966)). All these characterizations are of little utility in applications, unless $M \in \mathcal{Z}$. The class of *$\mathcal{Z}$-matrices* or *matrices of class $\mathcal{Z}$* is the class of square matrices $M = [a_{ij}]$, such that $a_{ij} \leqq 0, \ \forall i \neq j$. See Fiedler and Ptàk (1962). Following Schefold (1978), always with reference to a Leontief-Sraffa system with joint production, described by the pair of square matrices (A, B), a commodity is said to be *separately producible* if it is possible to produce a net output consisting of a unit of that commodity alone with a nonnegative intensity vector. In other words, commodity j is separately producible if and only if there exists a nonnegative vector x^j such that

$$(B - A)x^j = e^j,$$

where e^j is the unit vector of $\mathbb{R}^n$. A square technique (A, B) is called by Schefold (1978) *all-productive* if all commodities are separately producible, that is for every semipositive vector y there exists a semipositive vector x such that

$$(B - A)x = y,$$

that is condition (2) of Theorem 1 holds. It is easily proved that (2) of Theorem 1 is equivalent to the existence of a *unique* $x \geq [0]$ such that for every given $y \geq [0]$ the previous equation holds. See also Abad, Gasso and Torregrosa (2011), Fiedler (1981). Peris and Villar (1993) say that system (1) is *strongly solvable* if the above condition holds. Bidard and Erreygers (1998 a,b) speak of *adjustment property.* We have seen that this condition is equivalent to $(B - A)^{-1} \geq [0]$. In fact, Schefold (1978) calls "all-productive" the (square) technique (A, B) if $(B - A)^{-1} \geq [0]$. The technique (A, B) is called by Schefold (1978) *all-engaging* if $(B - A)^{-1} > [0]$.

For the reader's convenience, we prove the equivalence (1) $\iff$ (2) of Theorem 1, perhaps the more interesting from an economic point of view (in fact a characterization of all-productive systems, with $M = (B - A)$).

• **Proof of** (1) $\iff$ (2) **of Theorem 1 ("strong solvability").**

If $M^{-1} \geq [0]$ (that is M is nonsingular with a semipositive inverse), then $x = M^{-1}d$ and from $d \geq [0]$ we have $x \geq [0]$ (the case $x = [0]$, $d \geq [0]$ is excluded, otherwise M^{-1} would be singular, against the assumptions).

Conversely, first we prove that under condition (2) the matrix M is nonsingular. Let us suppose, absurdly, that $rank(M) = r$, with $r < n$, and suppose, without loss of generality, that the first r columns of M, $M^1, ..., M^r$ are linearly independent. Let $d \geq [0]$ and $M^1, ..., M^r, d$ be linearly independent. So, system $Mx = d$ has no solution, which is a contradiction. Now we prove that $M^{-1} \geq [0]$. Consider any vector $d \geq [0]$ such that $x = M^{-1}d$. Choose $d = e^i$, $i = 1, ..., n$. From $x \geq [0]$ we get $M^{-1} \geq [0]$. □

Bidard (1991, 2004) and Bidard and Erreygers (1998a,b) characterize all-engaging systems in the following way. See also Fiedler and Ptàk (1966). We recall that $M = (B - A) \in S$, i. e. $(B - A)$ is productive if there exists $x^0 \geq [0]$ such that $Mx^0 > [0]$. We have the following equivalence.

$$\left\{\exists x^0 \geq [0],\ Mx^0 \geq [0],\ \{(x \geq [0],\ Mx \geq [0]) \Longrightarrow x > [0]\}\right\} \iff M^{-1} > [0].$$

Similarly, the same authors prove the following characterization of all-productive systems:

$$\{M \in S,\ \{(x \geq [0],\ Mx > [0]) \Longrightarrow x > [0]\}\} \iff M^{-1} \geq [0].$$

See also Giorgi (2014), Giorgi and Magnani (1978), Villar (2003). The counterexample of Giorgi (2014) is not valid. Fujimoto, Silva and Villar (2003) give an extension of the above characterizations and call "essentiality condition" the above implications between brackets. From an economic point of view: the only way to obtain a semipositive (or a positive) net product, is to let all methods be operated.

It is quite obvious to expect that all-productive systems and a fortiori all-engaging systems have almost all properties of single production systems. This has been shown by Schefold (1978). We note, moreover, that if a system is all-productive, it is also productive, i. e. (2) holds, but the vice-versa does not hold. Indeed, if $(B-A)$ is all-productive, this means that for *every* $y \geq [0]$ there exists $x \geq [0]$ such that $(B-A)x = y$. It is sufficient to choose $y > [0]$. A fortiori if a model is all-engaging, it is also productive, but the vice-versa does not hold. In other words, the fact that $(B-A)$ is all-productive or all-engaging is a sufficient condition for productivity, but not a necessary one. Consider the following example taken from Giorgi and Magnani (1978).

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 3 & 6 \\ 2 & 1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 8 & 5 \\ 4 & 3 & 5 \end{bmatrix}.$$

It appears that $(B-A)$ is productive, but $(B-A)^{-1}$ is not semipositive.

As previously said, Bidard (1996) and Bidard and Erreygers (1998a,b) call *strongly viable* a system that satisfies condition (2), whereas a system that satisfies property (2) of Theorem 1 is called, by the same authors, an *adjustable system* or a system having the *adjustment property.*

Given a real square matrix M of order n, any representation of M in the form

$$M = B - A,$$

where A and B are square of the same order, is called a "splitting" or a "split". In particular, a splitting of a square matrix $M = B - A$, where $A \geqq [0]$ and $B \geqq [0]$, is called a *positive splitting.* There are various results, besides Theorem 1, on splittings of the said type. See, e. g. for a survey, Woznicki (1994, 2001). Most of these results require not only that $B \geqq [0]$, $A \geqq [0]$, but also that $B^{-1} \geq [0]$ and/or $A^{-1} \geq [0]$. However, in a Leontief-Sraffa model with joint production, this is a drawback, as a result of Johnson (1983) states that a semipositive square matrix has a semipositive inverse only if it is a diagonal matrix or a permutation of a diagonal matrix. See also Ding and Rhee (2014). Therefore, when it is required $B^{-1} \geq [0]$, this implies that joint-production is ruled out in this case. Similarly, the condition $A^{-1} \geq [0]$ has scarse economic meaning in this case.

Obviously, if $(B-A) \in \mathcal{Z}$, i. e. $b_{ij} \leqq a_{ij}$ $\forall i \neq j$ (terminology of Fiedler and Ptàk (1962)), we can apply to $(B-A)$ all the equivalent conditions to obtain that $(B-A)$ is all-productive (and in this case, also productive), such as the famous *Hawkins-Simon conditions.* We speak in this case of (nonsingular)

$\mathcal{M}$*-matrices* ($\mathcal{K}$*-matrices* in ths terminology of Fiedler and Ptàk (1962)). See, e. g. for a survey, Berman and Plemmons (1994), Giorgi (2022, 2023), Magnani and Meriggi (1981), Plemmons (1977), Poole and Boullion (1974). Obviously, $(B-A)$ can be all-engaging or all-productive with $(B-A) \notin \mathcal{Z}$, as it appears by choosing

$$A = \begin{bmatrix} 1.5 & 1 & 4 \\ 0 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 2 & 2 \\ 5 & 3 & 2.5 \end{bmatrix}.$$

The assumption $(B-A) \in \mathcal{Z}$ is made, for instance, by Fujimoto (1979) and by Bapat, Olesky and Van Den Driessche (1975). However, as admitted by Fujimoto (1979), this assumption says that there exists no net joint production; see also the critical comments of Peris and Villar (1993). Other assumptions on the pair (A, B), useful to extend the classical Perron-Frobenius theorem and to study linear joint production systems, are made by Mangasarian (1971). In the economic literature this approach was anticipated by Hicks (1965). See also the papers of Giorgi and Magnani (1978), Łos (1971), Punzo (1980), Fujimoto and Krause (1988), Mehrmann, Oleski, Phan and Van Den Driessche (1999).

Definition 3. A square matrix M is said to satisfy the *positive inclusion property* or *property of non dominant decomposition (NDD)* if it admits the following splitting:

$$\begin{aligned} M &= B - A,\ A \geqq [0]\,, \quad B \geqq [0]\,,\ B \text{ non singular, and} \\ Bx &\geqq [0] \Longrightarrow Ax \geqq [0]\,. \end{aligned}$$

This last condition simply says that no positive output is possible without consuming some inputs, which is obviously acceptable in all production models. Mangasarian (1971) introduced the above implication in his generalizations of the Perron-Frobenius theorem to the pair (A, B) (see Section 5 of the present paper) and proves that the above implication is equivalent to the existence of a matrix $H \geqq [0]$ such that

$$A = HB.$$

If B is non singular, as required by Definition 3, we obtain

$$H = AB^{-1} \geqq [0]\,.$$

Hence, if a matrix $M = B - A$ satisfies the (NDD) condiiton, it is possible to represent the same in the form

$$M = (I - H)B, \quad \text{with } H = AB^{-1} \geqq [0] \ \ (\text{and } A, B \geqq [0]\,).$$

Note that $(I - H) \in \mathcal{Z}$. Similarly, it can be proved that

$$y^\top B \geqq [0] \Longrightarrow y^\top A \geqq [0]$$

is equivalent to the existence of $H \geqq [0]$ such that

$$A = BH. \ \ (H = B^{-1}A, \text{ if } B \text{ is non singular}).$$

The implication

$$Ax \geqq [0] \Longrightarrow Bx \geqq [0]$$

is equivalent to the existence of $H \geqq [0]$ such that

$$B = HA. \quad (H = BA^{-1} \text{ if } A \text{ is nonsingular}).$$

The implication

$$y^{\top} A \geqq [0] \Longrightarrow y^{\top} B \geqq [0]$$

is equivalent to the existence of $H \geqq [0]$ such that

$$B = AH. \ \ (H = A^{-1}B \text{ if } A \text{ is nonsingular}).$$

We have the following result.

Theorem 2. Let $M = (B - A)$ be a square matrix satisfying the (NDD) property, that is

$$M = (I - AB^{-1})B, \quad A, B \geqq [0]\,, \ AB^{-1} \geqq [0]\,.$$

If $M^{-1} \geq [0]$, the following properties are satisfied.
(i) There exists $x \geqq [0]$ such that $Mx > [0]$.
(ii) $\left\{p \geqq [0]\,,\ p^{\top} M \leqq [0]\right\} \Longrightarrow p = [0]$.
(iii) $\rho(A_B) < 1$, where

$$\rho(A_B) = \max\left\{\lambda : \det(A - \lambda B) = 0\right\} \equiv \lambda^*(AB^{-1}),$$

where λ^* is the Frobenius eigenvalue of AB^{-1}; indeed we have

$$|A - \lambda B| = 0 \Longrightarrow |HB - \lambda B| = 0 \Longrightarrow |H - \lambda I| \cdot |B| = 0 \Longrightarrow \left|AB^{-1} - \lambda I\right| = 0.$$

(iv) AB^{-1} is convergent, i. e.

$$\lim_{n \longrightarrow +\infty} (AB^{-1})^n = [0]\,.$$

Proof.
(i) Obviously, this condition is satisfied: it is sufficient to choose $d > [0]$. We obtain

$$x = M^{-1}d \geqq [0] \quad \text{and} \quad Mx = d > [0]\,.$$

(ii) Let $p \geqq [0]$ such that $p^{\top} M \leqq [0]$. Multiplying by $M^{-1} \geq [0]$ we obtain that $p \leqq [0]$, and hence $p = [0]$.

Conditions (iii) and (iv) are equivalent and hence it is sufficient to prove that one of them holds true. The theorem of Mangasarian, applied to the present case, ensures the existence of a vector $v \geqq [0]$ such that

$$v^\top A = \lambda^* v^\top B, \quad \text{with } \lambda^* = \rho(A_B),$$

that is

$$v^\top M = v^\top (B - A) = (1 - \lambda^*) v^\top B.$$

Multiplying by M^{-1} we get

$$[0] \leqq v = (1 - \lambda^*) v^\top B A^{-1},$$

from which $\lambda^* < 1$. $\square$

Needless to say that a matrix $M = B - A$ can satisfy $(i) - (iv)$ of Theorem 2, without being $M^{-1} \geq [0]$. Take, for instance,

$$M = B - A = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} - \begin{bmatrix} 0 & 0.25 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 \\ 0.5 & 0.5 \end{bmatrix}.$$

It holds

$$AB^{-1} = \begin{bmatrix} 0.25 & 0.5 \\ 0 & 0.5 \end{bmatrix} \geqq [0]$$

and properties $(i) - (iv)$ of Theorem 2 are satisfied, yet M^{-1} is not semipositive.

Definition 4. A square matrix M is said to be a *B-matrix* or that it admits a *B-splitting* if

1. M satisfies the (NDD) property:

$$M = B - A, \quad A, B \geqq [0], \ B \text{ nonsingular}, \ AB^{-1} \geqq [0].$$

2.

$$\left\{ \begin{array}{c} Mx \geqq [0] \\ Bx \geqq [0] \end{array} \right\} \Longrightarrow x \geqq [0].$$

Condition 2 says that the matrix $J = \begin{bmatrix} M \\ B \end{bmatrix}$ is a *monotone matrix.* We have the following result (Peris (1991)).

Theorem 3. Let the square matrix M admit a B-spliting, i. e. $M = B - A$ is a B-matrix. Then, the following conditions are equivalent.

(1) M^{-1} exists and is semipositive: $M^{-1} \geq [0]$.

(2) There exists $x \geq [0]$ such that $Mx > [0]$ ("productivity condition").

(3) $\rho(A_B) < 1$, i. e. $\lambda^*(AB^{-1}) < 1$.

(4) AB^{-1} is convergent, i. e. $\lim_{n \longrightarrow +\infty} (AB^{-1})^n = [0]$.

(5) For all $d \geq [0]$ there exists $x \geq [0]$ such that $Mx = d$ (that is $Mx = d$ is "strongly solvable").

Proof. Recall that, under the assumption of the theorem, we have $M = (I - AB^{-1})B$, with $AB^{-1} \geqq [0]$, i. e. $(I - AB^{-1}) \in \mathcal{Z}$. Then we make reference to the theory of $\mathcal{M}$-matrices and to Perron-Frobenius theorem (see also Debreu and Herstein (1953)). The implication (1)$\Longrightarrow$(2) is obvious. If (2) holds, this means that there exists $x \geq [0]$ such that $Mx = y > [0]$. We know that

$$(I - AB^{-1})z > [0] \text{ for } z = Bx > [0].$$

As $(I - AB^{-1}) \in \mathcal{Z}$, this condition implies

$$\lambda^*(AB^{-1}) < 1,$$

i. e. condition (3). Now, let $\lambda^*(AB^{-1}) < 1$. Let $x \in \mathbb{R}^n$ such that $Mx \geqq [0]$. Then

$$Mx = (I - AB^{-1})Bx \geqq [0].$$

As $(I - AB^{-1})^{-1} \geq [0]$, by condition (3) (recall that $(I - AB^{-1}) \in \mathcal{Z}$), it follows

$$(I - AB^{-1})^{-1}Mx = Bx \geqq [0].$$

But, being M a B-matrix, we have $x \geqq [0]$. Then M is monotone and hence condition (1) holds. It is known from Theorem 1 that $M^{-1} \geq [0]$ is equivalent to (5) and that (3) is equivalent to (4). □

Note that, under the assumption of Theorem 3, we can say that the above conditions (1)-(5) are also equivalent to:

• The matrix $(I - AB^{-1})$ satisfies the *Hawkins-Simon conditions,* that is all its leading principal minors are positive. Indeed, being $(I - AB^{-1}) \in \mathcal{Z}$, we have $\lambda^*(AB^{-1}) < 1$ if and only if the Hawkins-Simon conditions hold. For a survey on these last conditions see Giorgi (2023).

Note that, under the assumptions of Theorem 3, we can say that the matrix $M = (B - A)$ is all-productive if and only if it is productive. Another result which links productivity of the pair (A, B) with all-productivity of the same, is given by Peris and Villar (1993) in Proposition 1 of the quoted paper.

The next theorem, again due to Peris (1991), characterizes inverse-positive matrices, i. e. square matrices having a semipositive inverse, in terms of B-splitting.

Theorem 4. Let be given a square matrix M. Then the following conditions are equivalent.

(a) M^{-1} exists and it holds $M^{-1} \geq [0]$.

(b) M admits a B-splitting, $M = B - A$ such that $\lambda^*(AB^{-1}) < 1$.

Proof.

(b)$\Longrightarrow$(a) is Theorem 3.

(a)$\Longrightarrow$(b). First note that in case M admits a semipositive inverse, then for any positive splitting of M, $M = (B - A)$, we have $\lambda^*(AB^{-1}) < 1$ (Theorem

1 of Peris (1991)). Moreover, $M \geqq [0]$ implies $x \geqq [0]$ and the second condition of B-splitting holds. Consider now a vector $y^\top = [1, 1, ..., 1]\, M^{-1} \geq [0]$ (really we have the strict inequality) and let

$$v = \frac{1}{\sum y_i + \theta} y \text{ for } \theta > 0.$$

We construct the matrix

$$T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \cdots & \cdots & \cdots & \cdots \\ v_1 & v_2 & \cdots & v_n \end{bmatrix} \geqq [0]\,,$$

For the matrix T we have $\lambda^*(T) = \sum v_i < 1$. Therefore, the $\mathcal{Z}$-matrix $(I - T)$ is invertible, with $(I - T)^{-1} \geq [0]$ and it holds

$$(I - T)^{-1} = I + T + (T)^2 + (T)^3 + ...$$

Furthermore

$$(I - T)^{-1} M = M + \frac{1}{\theta} E,$$

where

$$E = (e_{ij}), \quad e_{ij} = 1 \text{ for all } i, j.$$

Then choose $\theta > 0$ such that $\frac{1}{\theta} > \max |m_{ij}|$. For such θ we have $(I - T)^{-1} M \geqq [0]$. Now let

$$\begin{aligned} B &= (I - T)^{-1} M \\ A &= TB. \end{aligned}$$

Then $M = (I - T)B = B - TB = B - A$ is a positive splitting. By construction, $AB^{-1} = T \geqq [0]$. Then this produces a B-splitting of the matrix M. $\square$

Note that every $\mathcal{Z}$-matrix has a B-splitting and hence the existence of a B-splitting for a square matrix M is an extension of the concept of $\mathcal{Z}$-matrices, in the sense that any matrix $M \in \mathcal{Z}$ can be split in the form

$$M = (sI - A),$$

with $A \geqq [0]$. So, $B = sI$ and A provide a B-splitting for M and the condition $\lambda^*(AB^{-1}) < 1$ in the above case just means $\lambda^*(A) < s$, a necessary and sufficient condition for a $\mathcal{Z}$-matrix to have a semipositive inverse, i. e. to belong to the class of nonsingular $\mathcal{M}$-matrices, called by Fiedler and Ptàk (1962) "class of $\mathcal{K}$-matrices". See Berman and Plemmons (1994), Giorgi (2022), Magnani and Meriggi (1981), Plemmons (1977), Poole and Boullion (1974). The converse is not true. The following example is taken from Peris (1991), The matrix

$$M = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & 1 & -1 \\ -\frac{1}{2} & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}$$

admits a B-splitting, but it is not a $\mathcal{Z}$-matrix. Hence the class of square matrices having a semipositive inverse is strictly contained in the class of B-matrices, but has only a partial overlapping with the class of $\mathcal{Z}$-matrices.

The conditions established above, ensuring the strong solvability of the quantity system, allow as well to obtain the solvability of the associated price system, i. e. of a Sraffa system with joint production. The equilibrium price vector for this kind of systems can be defined through the following equation

$$p^\top B = (1 + r)p^\top A + w\ell^\top, \tag{3}$$

where $p \in \mathbb{R}^n$ stands for the price vector, $\ell > [0]$ is the vector of labour requirements and w and r are scalars representing the wage and profit rate, respectively, with $r \geqq 0$ and $w \geqq 0$. Assume that the matrix $M = (B - A)$ is all-productive, i. e. (Theorem 4) it admits a B-splitting, with $\lambda^*(AB^{-1}) < 1$. The above system can be written as

$$p^\top (B - A) = rp^\top A + w\ell^\top,$$

that is

$$p^\top = rp^\top A(B - A)^{-1} + w\ell^\top (B - A)^{-1}.$$

Hence, all properties of single product techniques with respect to the price system hold, since the Perron-Frobenius theorem and the theory of $\mathcal{M}$-matrices (see, e. g., Berman and Plemmons (1994), Giorgi (2022), Magnani and Meriggi (1981), Plemmons (1977)) can be applied to the nonnegative matrix $X \equiv A(B - A)^{-1}$. Indeed, we have

$$p^\top - rp^\top A(B - A)^{-1} = w\ell^\top (B - A)^{-1},$$

that is

$$p^\top (I - rA(B - A)^{-1}) = w\ell^\top (B - A)^{-1},$$

that is

$$p^\top = w\ell^\top (B - A)^{-1}(I - rA(B - A)^{-1})^{-1}$$

and, if $r > 0$, we have that

$$(I - rA(B - A)^{-1})^{-1}$$

exists and is semipositive if and only if $\frac{1}{r} > \lambda^*(X)$. Therefore, in this case, if also $w > 0$, we have $p > [0]$.

Now we present a result due to Bidard (2004), which puts into relationship all-productive systems with all-engaging systems. See also Bidard and Erreygers (1998b).

Definition 5. The economic (square) system (A, B) is *physically decomposable* or *physically reducible* if, after suitable permutations of the rows and columns of the matrices A, B, the same can be writtn as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ [0] & A_{22} \end{bmatrix}; \quad B = \begin{bmatrix} B_{11} & B_{12} \\ [0] & B_{22} \end{bmatrix},$$

where A_{11} and B_{11} are square matrices of the same order. If a system is not physically decomposable, it is *physically indecomposable* (or *physically irreducible*).

Note that the above definition does not coincide with the notion of "non-basic system" of Sraffa for joint production. See further and see Manara (1968). In the joint production case, a system may be (physically) indecomposable, all-productive but not all-engaging. Consider the following example.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We have

$$B - A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \quad (B - A)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Giorgi and Magnani (1978), Bidard (1996, 2004), Bidard and Erreygers (1998), Schefold (1978) introduce the notion of *g-all-productive systems* and *g-all-productive systems.*

Definition 6. Let $g > -1$ represent a rate of accumulation. The square system (A, B) is *g-all-productive* if

$$(B - (1 + g)A)^{-1} \geq [0]$$

and is *g-all-engaging* if

$$(B - (1 + g)A)^{-1} > [0] .$$

In other words, the square system (A, B) is a g-all-productive system if it can produce any final demand vector d after accumulation at the rate g. Obviously, an all-productive system is g-all-productive at $g = 0$, and similarly for all-engaging systems. The following theorem, due to Bidard (1996, 2004), gives a characterization of g-all-engaging systems.

Theorem 7. If (A, B) is g-all-engaging for some value g, the set S' on which it remains g-all-engaging is an interval of the type $S' = (g_{\min} = g_0, g_{\max} = G)$. The upper bound of S' is characterized by the properties

$$\begin{aligned} \exists! q &> [0] \text{ such that } [B - (1 + G)A]\, q = [0] , \\ \nexists y &\geq [0] \text{ such that } [B - (1 + G)A]\, y \geq [0] . \end{aligned}$$

The symbol $\exists!$ indicates the existence and uniqueness of activity levels q, up to a positive scalar.

A similar result has been proved by Schefold (1978). Theorem 7 shows that an all-engaging system remains all-engaging for $g > -1$ provided that g is smaller than the upper bound G. The number G is the maximal growth rate, which appears in the classical von Neumann model of economic growth. Note that in single production systems, where A is semipositive and indecomposable, the matrix $(I-(1+g)A)^{-1}$ is positive for any $g < G$, where $\frac{1}{1+G} = \lambda^*(A)$, the Frobenius eigenvalue of A.

Bidard and Erreygers (1998b) observe that in joint production systems the value G is a root of $\det(B-(1+g)A)=0$, but not necessarily the first positive root, as in the case of single production systems. The same authors note also that the existence of a value G such that the two conditions of Theorem 7 are met is not guaranteed. "In that case, the system cannot be all-engaging. Assume the existence of such a value. Then it can be shown that (A,B) is indeed g-all-engaging on some left neighbourhood of G ...[which]does not necessarily contain the value $g=0$. The system is all-engaging when, in terms of the set S', we have $g_0 < 0$" (Bidard and Erreygers (1998b), page 433).

Bidard (2004) in Theorem 6 of Chapter 12 studies the relationships between all-productive systems and all-engaging systems in terms of reducibility or irreducibility of (A,B).

Theorem 8. If (A,B) is all-productive but not all-engaging:

(i) either (A,B) is physically irreducible and 0 is the lower bound g_0 of the interval S' defined in Theorem 7, or

(ii) (A,B) is reducible, but the set on which (A,B) is g-all-productive is an interval $S=[g_0,G)$ which contains the value $g=0$.

The upper bound G of S has the two properties

$$\begin{aligned} \exists q &\geq [0]\,, \;\; [B-(1+G)A]\,q=[0] \\ \nexists y &\geq [0]\,, \;\; [B-(1+G)A]\,y>[0]\,. \end{aligned}$$

The existence of G satisfying the above two relationships does not guarantee that S is non-empty. These results of Bidard will be reconsidered in Section 5 of the present paper.

The matrix $(B-(1+g)A)$ is considered in models of the type

$$\begin{cases} \quad (B-(1+g)A)=d \\ x \geq [0]\,,\; g>0,\; d \geq [0]\,. \end{cases}$$

In other words, it is possible to get a *uniform* distribution of the net product. It must be observed (see Giorgi and Magnani (1978)) that there exist productive models of the type (A,B) which do not admit a uniform distribution of the net product. The same is true for profitable models, with reference to the uniform distribution of the net value. Consider, with regard to this subject, the following example (Giorgi and Magnani (1978)).

$$A=\begin{bmatrix} 3 & 3 & 1 \\ 1 & 3 & 6 \\ 2 & 1 & 1 \end{bmatrix}; \quad B=\begin{bmatrix} 1 & 0 & 2 \\ 3 & 8 & 5 \\ 4 & 3 & 5 \end{bmatrix}; \quad d=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Other results concerning the system

$$(1+g)Ax = Bx$$

are shown by Steenge (1986). This author in a first step assumes that the following implication (already considered) holds:

$$Bx \geqq [0] \implies Ax \geqq [0]\,,$$

i. e. there exists a nonnegative matrix H such that

$$A = HB,$$

and assuming that B is nonsingular, then we have

$$H = AB^{-1}.$$

This author proposes the following extension of the above conditions:
• There exist two nonsingular nonnegative matrices P and Q, such that $A = PHQ$ and $B = PQ$, where H is a semipositive productive, nonsingular and indecomposable matrix.

Note that the first conditions are formally obtained by putting $P = I$, which implies $B = Q$. Under these assumptions, the author proves (besides other results) that the characteristic equation $\det(B - \lambda A) = 0$ possesses a simple, real and positive root, which is the smallest of all real and positive roots if there are more than one.

Now we make some remarks on the notion of basic and non-basic systems proposed by Sraffa (1960) for joint production systems. In *Production of Commodities by Means of Commodities* Sraffa (1960) proposes a number of specific problems, that the author discusses without use of mathematics, which paradoxically makes the book rather difficult to read. One of these problems concerns the distinction between *basic* and *non-basic commodities,* namely in single production models, between commodities which enter directly or indirectly as means of production in all production processes of production (basic commodities) and commodities which enter only in a sub-system of the whole number of the production processes (non-basic commodities). It is well known that in single production models, i. e. $A \geqq [0]$ of order n, and $B = I$, the presence of all basic commodities is equivalent to the *irreducibility* (or *indecomposability)* of the square matrix A (see. e. g., Giorgi (2019), Varri (1979)). This condition is in turn, a necessary and sufficient condition to ensure the existence of pairs (λ, x) and (μ, p) which solve the systems

$$\begin{cases} Ax = \lambda x \\ x > [0]\,, \quad \lambda > 0, \end{cases}$$

$$\begin{cases} p^\top A = \mu p^\top \\ p > [0]\,, \quad \mu > 0. \end{cases}$$

In the said problems, then we have $\lambda = \mu = \lambda^*(A)$, the Frobenius root of A.

When treating joint production models, Sraffa changes his previous definitions concerning basic and non-basic commodities. This "new" definition was formalized in mathematical terms by Manara (1968), who requires the existence of permutation matrices P and Q and of a matrix H such that

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{22}H & A_{22} \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{22}H & B_{22} \end{bmatrix},$$

with A_{11} and B_{11} square sub-matrices of the same order, in which case the commodities referred to the rows of A_{22} are identified as non-basic.

This qualification depends on the following fact. Let n_1 and $n_2 = (n - n_1)$ be, respectively, the numbers of rows of A_{11} and A_{22}; take the transpose $P^\top$ of P and define the vector

$$p^\top P^\top = (p^1, p^2)^\top,$$

with p^1 of order n_1. Then the simple post-multiplication of both sides of the equation present in the system

$$\begin{cases} p^\top B = \mu p^\top A \\ p > [0], \quad \mu > 0 \end{cases} \tag{4}$$

by the nonsingular matrix

$$M = \begin{bmatrix} I_{n_1} & [0] \\ -H & I_{n_2} \end{bmatrix},$$

where I_{n_1} and I_{n_2} are identity matrices of order, respectively, n_1 and n_2, just gives the equivalent system

$$\begin{cases} (p^1)^\top \{B_{11} - B_{12}H - \mu(A_{11} - A_{12}H)\} = [0] \\ (p^2)^\top (B_{22} - \mu A_{22}) = (p^1)^\top (\mu A_{12} - B_{12}), \end{cases}$$

which appears to be decomposed into two sub-systems, where only the second one involves all the components of p. Therefore, there is some sort of one-to-one way dependence of the prices of the commodities associated with the last n_2 rows of PAQ and PBQ on the other commodities. We wish to stress that, contrary to what happens in the single production case, in joint production the absence of non-basic commodities (i. e. all commodities are basic, in the above Sraffa-Manara sense) is neither necessary nor sufficient to ensure the solvability, neither of system (4), nor of system

$$\begin{cases} Ax = \lambda Bx \\ x > [0], \quad \lambda > 0. \end{cases} \tag{5}$$

In order to prove that the above condition is not necessary, it is sufficient to choose

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 4/3 & 1 \\ 0 & 1 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & b_{12} & b_{13} \\ 0 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix},$$

with

$$\begin{aligned} a_{13} &\geqq 0,\ 0 < b_{13} - 2a_{13} \leqq 6a_{12}, \\ b_{12} &= 2a_{12} + (2a_{13} - b_{13})/3, \end{aligned}$$

in which case A and B have all semipositive lines, the model is productive and profitable and systems (4) and (5) admit solutions (and only these solutions) described by

$$\begin{aligned} \lambda &= \mu = 2,\ \ x_1 > 0,\ x_2 = 3x_3,\ x_3 > 0, \\ p^\top &= (p_1; p_1(b_{13} - 2a_{13}); p_3),\ p_1 > 0,\ p_3 > 0, \end{aligned}$$

although commodities 2 and 3 are non-basic, as PAQ and PBQ show, with $P = Q = I$, $n_1 = 1$, $H = [0]$.

The proof that the same condition is not suficient to make (4) and (5) solvable may be obtained by taking into account the well known Manara's example:

$$A = \begin{bmatrix} 1 & 1.1 \\ 1.1 & 1 \end{bmatrix};\ \ B = \begin{bmatrix} 1.09 & 1.144 \\ 1.144 & 0.99 \end{bmatrix},$$

where both systems (4) and (5) are not solvable, even if the positive matrices A and B are elements of a model which is productive, profitable and without non-basic commodities. Therefore our thesis is proved.

One may ask if there are relationships between the Sraffa-Manara definition of a non-basic system and the usual definition of a square reducible matrix. The question has been investigated by Steedman (1977, 1980) and Pasinetti (1980). However, see also Schefold (1971) and Abraham-Frois and Berrebi (1976) (French original edition of the book published in English in 1979). Steedman (1977, 1980) considers another definition of basic and non-basic commodities in a square model (A, B), by means of the matrix

$$H = A(B - A)^{-1}$$

obviously under the assumption that $(B-A)$ is nonsingular. It must be observed that the above matrix H was previously considered by Pasinetti (1973), and called by this author "direct and indirect capital matrix". The main result of Steedman is that the (square) system (A, B) is reducible in the sense of Sraffa-Manara if and only if H is reducible in the usual sense.

Also Pasinetti (1980) is concerned with the relationships betwen the Sraffa-Manara definition of reducibility and the notion of reducibility in the usual sense. Assuming that B is nonsingular, Pasinetti proves that the system (A, B) is reducible in the sense of Sraffa-Manara if and only if AB^{-1} is reducible in the usual sense. Also this result was anticipated by Schefold (1971) and by Abraham-Frois and Berrebi (1976).

It must be remarked that the notion of reducibility in the Sraffa-Manara sense can be extended to the case of *non necessarily square systems* (A, B), whereas this is not directly possible by adopting the notions of Steedman and Pasinetti. For another definition of basic and non-basic commodities in a square system (A, B), see Flaschel (1982) and Sanchez Choliz (1992).

Another paper in which the mathematical properties of square joint production systems are analyzed is Woods(1987). This author relies his analysis mainly on known theorems of the alternative for linear systems. A similar approach is considered by L. and C. Filippini (1982). Finally, an interesting result of Peris (1991) states that if the square matrix M has a strictly positive column, then its inverse, when it exists, is not semipositive (if M has a strictly positive row, we can reason with $M^\top$ in order to obtain the same conclusion). This remarkable property was proved also by Johnson (1983) by means of the analysis of matrix sign patterns. Really, Theorem 7 of Peris (1991) is more general than the above assertion.

4. Some extensions to non square systems

In the present section we remove the assumption of equality between the number of sectors and the number of commodities, i. e. we consider non square joint productions systems. We consider again an economy described by the system

$$Bx = Ax + d \tag{6}$$

where A and B are semipositive matrices of order (m, n), i. e. n different production processes are available to produce m goods. All processes operate under constant returns to scale. For the reader's convenience, we recall the basic notions already anticipated in Section 2. Production process j $(j = 1, ..., n)$ is described by the j-th column vector of inputs $A^j \in \mathbb{R}^m_+$ and obtains the outputs described by the corresponding j-th column $B^j \in \mathbb{R}^m_+$. The activity level of process $j = 1, ..., n$, is described by $x_j \geqq 0$. For the economy as a whole the (m, n) matrix A represents the n input vectors and the (m, n) matrix B represents the n output vectors. We assume that A has all semipositive columns and that B has all semipositive rows. See Kemeny, Morgenstern and Thompson (1956). We recall also the following notions, already anticipated.

• The model (A, B) is *productive* if $(B - A) \in \mathcal{S}$, i. e. the following system

$$\begin{cases} (B - A)x > [0] \\ x \geq [0]\,, \text{ (equivalently: } x > [0]\,) \end{cases}$$

admits a solution.

• The model (A, B) is *viable* (or *quasi-productive*) if $(B - A) \in \mathcal{S}_0$, i. e. the following system

$$\begin{cases} (B - A)x \geqq [0] \\ x \geq [0] \end{cases}$$

admits a solution.

• The model is *profitable,* if $(B-A)^\top$ is productive, i, e, the following system

$$\begin{cases} y^\top(B-A) > [0] \\ y \geq [0]\,,\ \text{(equivalently: } y > [0]\,) \end{cases}$$

admits a solution. In other words, $(B-A) \in \mathcal{S}^\top$.

It must be noted that, by Theorem 3.8 of Fiedler and Ptàk (1966), if $(B-A) \in \mathcal{S}_0$ and moreover,

$$\{x \geq [0]\,,\ (B-A)x \geqq [0]\} \Longrightarrow x > [0]\,,$$

then $m \geqq n$.

We have already remarked that in linear (not necessarily square) joint production models the two properties of productivity and profitability are compatible, but independent properties. In other words, the class $\mathcal{S}$ and $\mathcal{S}^\top$ are not disjoint, but have only a partial overlapping. An exception is given by the case of A and B square, of order n, and $B = I$, i. e. the case of (square) single production models. In this case $(B-A) = (I-A) \in \mathcal{Z}$ and, thanks to the closure property of the class of $\mathcal{M}$-matrices with respect to transposition, the model (A, I) is productive if and only if it is profitable. In the general case we can formulate the following productivity and profitability characterizations for a general model (A, B). We need a previous result.

Lemma 1. (Ville theorem of the alternative). Let A be a real matrix of order (m, n). The system

$$Ax > [0]\,,\ x > [0]$$

has a solution if and only if the system

$$y^\top A \leqq [0]\,,\quad y \geq [0]$$

has no solution.

See, e. g. Mangasarian (1969). In terms of the classes $\mathcal{S}$ and $\mathcal{S}_0$, the Ville theorem of the alternative can therefore be described by the following equivalence:

$$(A \in \mathcal{S}) \iff ((-A)^\top \notin \mathcal{S}_0).$$

Theorem 9. Let A and B be, respectively, the input and the output matrix of an economic linear model involving m goods and n processes. Then:

(i) The model (A, B) is productive if and only if, for any price vector $y \geq [0]$, there exists an activity (in general varying with the choice of y) such that the corresponding net value is positive:

$$y \geq [0] \Longrightarrow \exists j : y^\top(B-A)^j > 0.$$

(ii) The model is profitable if and only if, for any activity vector $x \geq [0]$, there exists a good (in general varying with the choice of x) such that the corresponding net production is positive:

$$x \geq [0] \Longrightarrow \exists i : (B-A)_i x > 0.$$

Proof. Thanks to the Ville theorem of the alternative, $(B-A) \in \mathcal{S}$ if and only if $\left[-(B-A)^\top\right] \notin \mathcal{S}_0$. This means that the system

$$\begin{cases} \left[-(B-A)^\top\right] y \geqq [0] \\ y \geq [0] \end{cases}$$

i. e. the system

$$\begin{cases} y^\top (B-A) \leqq [0] \\ y \geq [0] \end{cases}$$

has no solution. Therefore (i) is proved. In a symmetric way, $(B-A)$ is profitable if and only if $(B-A)^\top \in \mathcal{S}$, i. e, thanks to the same theorem of the alternative, if and only if $\left[-(B-A)^\top\right]^\top \notin \mathcal{S}_0$, i. e. $[-(B-A)] \in \mathcal{S}_0$. This means that the system

$$\begin{cases} -(B-A)x \geqq [0] \\ x \geq [0] \end{cases}$$

i. e. the system

$$\begin{cases} (B-A)x \leqq [0] \\ x \geq [0] \end{cases}$$

has no solution. Therefore (ii) is proved. □

Obviously, the practical relevance of the above tests relies on the possibility to detect non productive models and non profitable models, rather than productive models or profitable models.

Another approach to study system (6) is to make reference to a suitable notion of *monotonicity* for $M = (B-A)$ and of *positive inclusion* for the pair (A, B), as suggested by Peris and Subiza (1992). The definition of monotonicity for a non square matrix M is formally the same of the square case: a matrix M, of order (m, n) is called *(rectangular) monotone* if

$$x \in \mathbb{R}^n,\ Mx \geqq [0] \Longrightarrow x \geqq [0].$$

Some generalizations of the above definition can be found in Berman and Plemmons (1976, 1994), Mangasarian (1968), Neumann, Poole and Werner (1982), Poole and Barker (1979). On this subject it is interesting also the previously quoted paper of Fiedler and Ptàk (1966). The next definition is taken from Peris and Subiza (1992).

Definition 7. A matrix M of order (m, n) is said to be *weak-monotone* if, for all $x \in \mathbb{R}^n$ and such that $Mx \geqq [0]$, there exists $y \geqq [0]$ such that $Mx = My$.

Note that any monotone matrix is weak-monotone and if $rank(M) = n$, the two definitions coincide. The general interest of weak-monotone matrices comes from the equivalence between the weak-monotonicity of M and the strong positive solvability of system (6), where $d \in Im(M) \cap \mathbb{R}_+^m$, being $M = B - A$ and $Im(M)$ stands for the image of $(B-A)$. Indeed, in the case examined in the present section, not all vectors in $\mathbb{R}_+^m$ are necessarily in the image of $(B - A)$.

Definition 8. We say that system (6) is *strongly solvable,* if, for any $d \geqq [0]$, $d \in Im(B - A)$, there exists $x \geqq [0]$, such that (6) holds.

Theorem 10. Let M be a matrix of order (m, n). Then the following conditions are equivalent:

(i) For any $d \geqq [0]$, $d \in Im(M)$, there exists $x \geqq [0]$ such that $Mx = d$.

(ii) If $Mx \geqq [0]$, there exists $y \geqq [0]$ such that $Mx = My$, i. e. M is weak-monotone.

(iii) If $Mx \geqq [0]$, then $x \in \mathbb{R}_+^n + \ker(M)$.

Proof. The chain of the implications will be:

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (i).$$

$(i) \Longrightarrow (ii)$. Let be $x \in \mathbb{R}^n$ such that $Mx \geqq [0]$. By denoting

$$d = Mx,$$

we have

$$d \geqq [0], \ d \in Im(M),$$

hence there exists $y \geqq [0]$ such that $My = d = Mx$.

$(ii) \Longrightarrow (iii)$. If $Mx \geqq [0]$, by relation (ii), there exists $y \geqq [0]$, such that $Mx = My$, from which

$$x - y \in \ker(M)$$

and

$$x = y + (x - y) \in \mathbb{R}_+^n + \ker(M).$$

$(iii) \Longrightarrow (i)$. Let be

$$d \in Im(M), \ d \geqq [0].$$

Then, there exists $x \in \mathbb{R}^n$ such that $Mx = d$. By means of condition (iii) :

$$x = u + v, \ u \in \mathbb{R}_+^n, \ v \in \ker(M).$$

But

$$d = Mx = M(x + v) = Mu,$$

from which

$$d = Mu, \text{ with } u \geqq [0]. \ \square$$

Also the notion of B-splitting, given in the previous section, can be extended to the non square case.

Definition 9. A positive splitting of an (m, n) matrix M, i. e.

$$M = B - A, \quad B \geqq [0], \quad A \geqq [0]$$

is said to be a *generalized B-splitting* of M if
(i) $Bx \geqq [0] \Longrightarrow Ax \geqq [0]$.
(ii) $\begin{pmatrix} M \\ B \end{pmatrix} x \geqq [0] \Longrightarrow \exists y \geqq [0]$ such that $Mx = My$.

Condition (ii) indicates that M is weak-monotone on the set

$$S = \{x \in \mathbb{R}^n : Bx \geqq [0]\}.$$

In Mangasarian (1971) it is shown that (i) of Definition 9 is equivalent to the existence of a square matrix $T \geqq [0]$ such that $A = TB$. Condition (i) may be called, as before, "positive inclusion" property. This fact allows to express M as the product of two matrices, i. e.

$$M = (B - A) = (I - T)B,$$

where $(I - T) \in \mathcal{Z}$ and $B \geqq [0]$. The following result follows (see Peris and Subiza (1992)).

Theorem 11. Let $M = B - A$ be a generalized B-splitting, with M of order (m, n) and assume that $rank(M) = m$ (this means that $Im(M) \cap int(\mathbb{R}^m_+) \neq \varnothing$). Then the following conditions are equivalent.
(a) M is weak-monotone, i. e. system (6) is strongly solvable.
(b) There exists $x \geqq [0]$ such that $Mx > [0]$, i. e. $M = B - A$ is productive.
(c) $\lambda^*(T) < 1$.

Proof.
$(a) \Longrightarrow (b)$. Since a positive vector $d \in Im(M)$ exists,

$$d = My > [0] \quad \text{for some } y \in \mathbb{R}^n.$$

Then, weak-monotonicity implies that there exists $x \geqq [0]$ such that

$$Mx = d > [0].$$

$(b) \Longrightarrow (c)$. By assumption there is some $x \geqq [0]$ such that

$$Mx = (I - T)Bx > [0].$$

As $(I - T) \in \mathcal{Z}$ and $Bx \geqq [0]$, this condition implies $\lambda^*(T) < 1$.
$(c) \Longrightarrow (a)$. Let be $x \in \mathbb{R}^n$, with $Mx \geqq [0]$. Then $(I - T)Bx \geqq [0]$, and, as $(I - T) \in \mathcal{Z}$, with $\lambda^*(T) < 1$, $I - T$ is monotone, which implies $Bx \geqq [0]$. Finally,

by condition (ii) of Definition 9, a vector $y \geqq [0]$ exists, such that $Mx = My$ and consequently, M is weak-monotone. $\square$

The following result, due to Peris and Subiza (1992), provides the relationships between weak-monotonicity and the existence of a generalized B-splitting for a matrix M (of order (m, n)). Here it is necessary to introduce an additional condition on the kernel of M.

Theorem 12. Let M be a matrix of order (m, n) such that $\ker(M) \cap \mathbb{R}^n_+ = \{[0]\}$. Then the following conditions are equivalent:

(a) M is weak-monotone.

(b) M allows a generalized B-splitting:

$$M = B - A,\ A = TB,\ \lambda^*(T) < 1.$$

It is also possible to characterize weak-monotonicity of M in terms of its generalized inverse. See Peris and Subiza (1992).

Another interesting result concerning non square joint production systems is given by Bidard and Erreygers (1998 a, b). We first introduce the following notation. Let $N = \{1, ..., n\}$ be the sets of the available processes in the economic system (A, B), with A and B of order (m, n) and both (at least) semipositive. Given a subset N_α of N, the technique denoted by (A_α, B_α) is the system in which the set of the available processes is restricted to N_α. Then we give the notion of *minimality* for a system (A, B).

Definition 10. The system (A, B) is *minimal* if it is productive (or strictly viable) and none of its techniques (A_α, B_α) is productive.

All-engaging and all-productive (square) systems are minimal. This can be deduced from the following result, due to Bidard and Erreygers (1998 a, b).

• If a system (A, B), with $A \geq [0]$, $B \geq [0]$, A and B of order (m, n), has any two of the following three properties:

(i) It is minimal;

(ii) It is square;

(iii) It is strongly solvable (in the terminology of Bidard and Erreygers, it is "adjustable"),

then it has also the third one.

For other results on minimal systems the reader is referred to Bidard and Errygers (1998 a,b). On the ground of the previous results and of the ones of Fiedler and Ptàk (1966), it seems that, the square linear joint production models, *whenever described by equalities,* are more interesting than the non square ones. Remember that the classical von Neumann model and, e. g., the modern economic description of the Marxian models, made by Morishima (1974, 1976), are represented by *linear inequalities,* involving non necessarily square matrices. See also the last section of Giorgi and Magnani (1978); in this last paper the matrix B which appears at page 460 must be corrected as

$$B = \begin{bmatrix} 6 & 3 \\ 1 & 12 \end{bmatrix}.$$

For an analysis of the relations between the von Neumann growth model and the linear joint production models of Sraffa, see the paper of Schefold (1980).

Remark 1. Always with reference to non necessarily square systems, Steenge and Konijin (1992) establish a link between a suitable "positive inclusion property" in the sense of Mangasarian (1971) and the definition of irreducibility of the pair (A, B) in the sense of Gale (1960). This notion is useful in establishing the equality between the maximal growth factor and the minimal interest factor in the classical von Neumann economic model. See, e. g., Gale (1960), Giorgi and Bidard (2025), Giorgi and Meriggi (1987, 1988), Murata (1977). Steenge and Konjin (1992) assume the following implication:

$$Bx > [0] \Longrightarrow Ax > [0]\,, \tag{7}$$

which is equivalent (if $Bx > [0]$ has a solution x) to

$$\exists T \geq [0]\,,\ \text{with each row is semipositive, such that } A = TB.$$

The system (A, B) is *reducible in the sense of Gale* if there exist two permutation matrices P, of order m, and Q, of order n, such that

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ [0] & A_{22} \end{bmatrix};\ \ PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where each row of B_{11} contains at least one positive element. Otherwise, the system is *irreducible in the sense of Gale.*

One of the results of Steenge and Konjin (1992) is the following one.

• Let the implication (7) be verified. Hence $A = TB$, with T having the above properties and let T be irreducible in the usual sense. Then the system (A, B) is irreducible in the sense of Gale.

As a corollary, the above authors obtain a link between irreducibility in the sense of Gale and the von Neumann original assumption $(A + B) > [0]$:

• Let the implication (7) be satisfied, Then if $(A + B) > [0]$ the system (A, B) is irreducible in the sense of Gale.

Indeed, the von Neumann condition alone (i. e. $(A + B) > [0]$) does not imply irreducibility. Robinson (1973) gives the following example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix};\ \ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This system is reducible, yet it satisfies the von Neumann condition.

Concerning the "dual system", i. e. the price system

$$p^\top B = (1+r)p^\top A + w\ell^\top, \tag{8}$$

with $r \geqq 0$, $w \geqq 0$, $\ell \geqq [0]$, we can obtain some results of solvability of the same when A and B are of order (m, n). We need a previous result, proved by Peris and Subiza (1992).

Lemma 2. Let M be an (m, n) matrix. Then the system $Mx = d$ is strongly solvable if and only if the system $M^\top z = c$ is strongly solvable. ($M^\top$ stands, as usual, for the transpose of M). In other words, M is weak-monotone if and only if $M^\top$ is weak-monotone.

Proof. We shall prove only one implication, as the other one follows in a similar way. Suppose that M is weak-monotone. If $M^\top$ is not weak-monotone, there exists $x \in \mathbb{R}^m$ such that $c = M^\top x \geqq [0]$ and the system

$$M^\top y = c,\ y \geqq [0]$$

has no solution. By Farkas' lemma (see, e. g., Gale (1960), Mangasarian (1969)), there will exist some vector $z \in \mathbb{R}^n$ such that

$$Mz \leqq [0]\,, \quad c^\top z > 0.$$

Now, as $M(-z) \geqq [0]$ and M is weak-monotone, there exists $u \geqq [0]$ such that $Mu = M(-z)$. Then

$$M(z+u) = [0] \text{ and } (z+u) \in \ker(M).$$

Since $\ker(M)$ and $Im(M^\top)$ are orthogonal supplementary subspaces,

$$c^\top (z+u) = 0,$$

and hence $c^\top z = -c^\top u \leqq 0$, because $c \geqq [0]$, $u \geqq [0]$, which contradicts the inequality $c^\top z > 0$. Therefore $M^\top$ is weak-monotone. □

We can remark that the system $M^\top y = c$, $y \geqq [0]$, when $M^\top$ is given by the difference of two nonnegative matrices, can be considered a "dual system", with respect to prices, of the system $Mx = d$, which considers the quantities of production.

Then we can prove the following result.

Theorem 13. Let be given system (8) and assume that $(B - A)$ is productive, that $Im(M) \cap int(\mathbb{R}^m_+) \neq \varnothing$, and that $(B-A)$ describes a generalized B-splitting, i. e.

$$\begin{aligned} Bx &\geqq [0] \Longrightarrow Ax \geqq [0]\,, \\ \begin{pmatrix} M \\ B \end{pmatrix} x &\geqq [0] \Longrightarrow \exists y \geqq [0] \text{ such that } Mx = My. \end{aligned}$$

Then, there exists a scalar $R > 0$ such that for all $r \in [0, R)$ the system (8) has a solution $p \geqq [0]$.

Proof. Define $M(r) = B - (1 + r)A$. By the productivity assumption, we have that there exists $\bar{x} \in \mathbb{R}^n_+$ such that $B\bar{x} > A\bar{x}$. Then let be

$$R = \sup \{r \geqq 0 : B\bar{x} > (1 + r)A\bar{x}\}.$$

Clearly, $R > 0$ and for any $r \in [0, R)$ Theorem 11 ensures that system

$$M(r)x = d$$

is strongly solvable. Then, by applying Lemma 2 we obtain the result. □

For other results on joint production price systems, see Fujimoto and Krause (1988).

5. Some hints on Perron-Frobenius results for pairs of matrices

There are in the literature several results concerning *nonlinear generalizations* of the celebrated Perron-Frobenius theorem, but curiously, there are relatively few results concerning the generalization of Perron-Frobenius theorem to pairs of matrices (A, B) or "matrix pencils", where A and B are not necessarily square and not necessarily both nonnegative. We give only some hints on the main results appeared in the literature and concerning the subject of the present section. In other words, instead of the equation

$$Ax = \lambda x, \ \lambda \geqq 0, \ x \geq [0],$$

where $A \geqq [0]$ of order n, in the present section it is considered the equation

$$Ax = \lambda Bx, \tag{9}$$

usually with x semipositive vector of $\mathbb{R}^n$, $\lambda \geqq 0$ and A, B matrices of order (m, n), matrices not necessarily semipositive. The scalar λ and the vector x of the pair (λ, x), solution of (9) are said, respectively, *generalized eigenvalue* and *generalized eigenvector of the pair of matrices* (A, B). Obviously, in linear economic models, it makes sense to consider A and B both (at least) semipositive.

A) The main results of O. L. Mangasarian (1971).

This author considers two matrices A and B, of order (m, n) and not necessarily nonnegative. Following Mangasarian (1971), we say that the complex number λ is an *eigenvalue of* A *relative to* B if (9) holds for some nonzero vector x; this vector $x \neq [0]$ is said to be an *eigenvector of* A *relative to* B

(corresponding to λ). The set of all eigenvalues of A relative to B is called the *spectrum of A relative to B* and denoted by

$$sp(A_B).$$

The *spectral radius of A relative to B* is defined by Mangasarian as follows:

$$\rho(A_B) = \begin{cases} \sup\limits_{\lambda \in sp(A_B)} |\lambda|\,, \text{ if } sp(A_B) \neq \varnothing \\ -\infty, \text{ if } sp(A_B) = \varnothing. \end{cases}$$

(The case of $sp(A_B) = \varnothing$ is of little interest and it is included for completeness).

Mangasarian proves that:

1) If $n > m$, every real λ belongs to $sp(A_B)$ and hence $\rho(A_B) = +\infty$.

2) If $n < m$, every real λ is in $sp(A^\top_{B^\top})$ and hence $\rho(A^\top_{B^\top}) = +\infty$.

3) If $m = n$, $sp(A_B)$ coincides with $sp(A^\top_{B^\top})$ and, when nonempty, it contains at most n numbers.

The above results enable to consider as interesting the problem $Ax = \lambda Bx$, if $n < m$; if $n > m$ only the problem $A^\top y = \lambda B^\top y$ is of interest, whereas if $n = m$, then both problems are interesting.

In order to formulate and discuss a Frobenius problem of the type

$$\begin{cases} Ax = \lambda Bx \\ x \geq [0]\,,\ \lambda > 0, \end{cases}$$

with A and B of order (m, n), Mangasarian makes use of the following lemma. Note that A and B are not assumed to be nonnegative.

Lemma 3. Let be given the matrices A and B, of order (m, n). Then the following conditions are equivalent:

(1)

$$y^\top B \geqq [0] \Longrightarrow y^\top A \geqq [0]\,.$$

(2) There exists a square matrix $T \geqq [0]$ such that

$$A = BT.$$

Under the assumption that $\{y : y^\top B \geq [0]\} \neq \varnothing$,the following conditions are equivalent:

(3)

$$y^\top B \geqq [0] \Longrightarrow y^\top A > [0]\,.$$

(4) There exists a square matrix $T > [0]$ such that

$$A = BT.$$

(Mangasarian takes into consideration also other “intermediate cases”. See Section 3 of the present paper).

By means of the said results Mangasarian obtains various Frobenius-type theorems for the pair (A, B). We give only some results.

Theorem 14. Let A and B be of order (m, n) and such that

$$y^\top B \geqq [0] \Longrightarrow y^\top A \geqq [0] .$$

Then:
(a) The system

$$\begin{cases} Ax = \lambda Bx \\ x \geq [0] \end{cases}$$

has a solution x for some $\lambda \geqq 0$. If, in addition $rank(A) = n$ or $rank(B) = n$, then $\lambda = \rho(A_B)$.
(b) If $rank(A)$ or $rank(B) = n$, then the system

$$\begin{cases} y^\top A = \lambda y^\top B \\ y^\top B \geq [0] \end{cases}$$

has a solution y for some $\lambda \geqq 0$. If, in addition, $m = n$, then

$$\lambda = \rho(A_B) = \rho(A^\top_{B^\top}).$$

From these results we can obtain the following corollary.

Corollary 1. Let A and B be square, of order n, and let the implication of Theorem 14 hold. Let B be nonsingular Then
(i)

$$\rho(A_B) = \rho(A^\top_{B^\top}) = \lambda^*(AB^{-1}).$$

(ii) The problem

$$\begin{cases} Ax = \lambda Bx, \\ x \geq [0] \quad \lambda \geqq 0 \end{cases}$$

is equivalent to the problem

$$(B^{-1}A)x = \lambda x, \;\; \lambda \geqq 0, \quad x \geq [0] .$$

Theorem 15. (Theorem of Frobenius-Mangasarian in a strong version). Let A and B be of order (m, n) and let be nonempty the set

$$\left\{ y : y^\top B \geq [0] \right\} .$$

Furthermore the following implication holds true:

$$y^\top B \geq [0] \Longrightarrow y^\top A > [0] .$$

Then:

(i) The problem

$$\begin{cases} Ax = \lambda Bx \\ x > [0] \end{cases}$$

has a solution x for some $\lambda > 0$.

(ii) If, in addition $rank(A) = n$ or $rank(B) = n$, then $\lambda = \rho(A_B) > 0$.

(iii) If $rank(A) = n$ or $rank(B) = n$, then the problem

$$\begin{cases} y^\top A = \mu y^\top B \\ \mu > 0, \ y^\top B > [0] \end{cases}$$

has a solution for some $\mu > 0$. If, in addition, A and B are square, it holds

$$\lambda = \mu = \rho(A_B) = \rho(A^\top_{B^\top}) > 0.$$

Note that if A, B are both square and, furthermore, the implication

$$y^\top B \geq [0] \Longrightarrow y^\top A > [0]$$

holds true, this is equivalent to the existence of a square matrix $T > [0]$ such that

$$A = BT.$$

If B is nonsingular, we have

$$B^{-1}A = T \geqq [0]$$

and therefore we can study the problem

$$\begin{cases} B^{-1}Ax = \lambda x \\ x > [0], \ \lambda > 0, \end{cases}$$

i. e. a usual Perron-Frobenius problem.

B) The main results of T. Fujimoto (1977).

This author assumes explicitly that A and B are *nonnegative square matrices* of order n. He remarks that, even under the said assumptions, Mangasarians's theorems do not cover some simple but economically significant cases, for example, the case where

$$B = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0.5 \\ 0.2 & 0.1 \end{bmatrix}.$$

On the other hand, the above case satisfies the assumptions A.1)-A.3) of Fujimoto (1977), given below. This author adopts the following symbology.

e : vector of $\mathbb{R}^n$ whose elements are all unity, i. e.

$$e^\top = [1, 1, ..., 1].$$

$S = \left\{x \in \mathbb{R}^n : x \geq [0]\,,\ e^\top x = 1\right\}$, i. e. S is the *fundamental simplex.*
$C(\lambda) = \{x : x \in S,\ \lambda Bx \geqq Ax\}$.
$C^+(\lambda) = \{x : x \in C(\lambda),\ x > [0]\}$.
$D(\lambda) = \{x : x \in S,\ \lambda Bx > Ax\}$.
$E(\lambda) = \{x : x \in S,\ \lambda Bx \geq Ax\}$.
Then Fujimoto makes the following assumptions.
$(A.1)$ There exists $x \geqq [0]$ such that

$$Bx > Ax$$

(i. e. $(B - A)x > [0]\,,\ x \geqq [0]$, i. e. the model is *productive*).
$(A.2)$ The matrix A is indecomposable in the usual sense.
$(A.3)$ $b_{ij} \leqq a_{ij},\ \forall i \neq j$, i. e. $(B - A) \in \mathcal{Z}$.

Note that $(A.1)$ and $(A.3)$ characterize the fact that $(B - A)$ is a matrix of class $\mathcal{M}$ and that, from an economic point of view, the fact that $(B - A) \in \mathcal{Z}$ implies that there is no net joint product. For properties of matrices of class $\mathcal{M}$ the reader is referred to Fiedler and Ptàk (1962), Giorgi (2022), Berman and Plemmons (1994), Magnani and Meriggi (1981), Plemmons (1977), Poole and Boullion (1974). Under $(A.1)$-$(A.3)$, Fujimoto (1977) proves the following results.

Theorem 16. Let $(A.1)$, $(A.2)$, $(A.3)$ be verified. Then:
(i) There is a real scalar λ^* such that $0 < \lambda^* < 1$, $C(\lambda) = \varnothing$ for $\lambda < \lambda^*$ and $C(\lambda) \neq \varnothing$ for $\lambda \geqq \lambda^*$.
(ii) $C(\lambda) = C^+(\lambda)$ for $\lambda < 1$.
(iii) If $E(\lambda) \neq \varnothing$ for some $\lambda < 1$, then $D(\lambda) \neq \varnothing$.
(iv) If there is an x^0 such that $\lambda^0 Bx^0 = Ax^0$, $x \in C(\lambda^0)$ for some $\lambda^0 < 1$, then there exists no x^1 such that $x^1 \neq x^0$, $\lambda_1 Bx^1 = Ax^1$, and $x^1 \in C(\lambda_1)$ for $\lambda_1 \geqq \lambda^0$.
(v) There exist a positive scalar λ^* and a positive vector x^* such that $\lambda^* Bx^* = Ax^*$; there is no $x \geqq [0]$ such that $x \neq kx^*$ for any scalar k and $\lambda^* Bx = Ax$; if $\lambda \neq \lambda^*$, then there is no $x \geqq [0]$ such that $\lambda Bx = Ax$.

Fujimoto (1977) gives also results for the generalized resolvent problem.

Theorem 17. Assume $(A.1)$, $(A.2)$, $(A.3)$ be verified. Then:
(i) The resolvent equation $\lambda Bx - Ax = d$, for any semipositive vector d, has a positive solution x^0 if $1 > \lambda > \lambda^*$ (λ^* being the positive scalar appearing in Theorem 16).
(ii) Given $d \geqq [0]$, there is no nonnegative solution other than x^0.
(iii) If $\lambda \leqq \lambda^*$, the resolvent equatrion has no nonnegative solution.

For a generalization of the above results to the nonlinear case, i. e. to the problem

$$F(x, \lambda) = [0]\,,$$

with $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$, the reader is referred to Fujimoto (1979, 1980).

C) The main results of Bapat, Olesky and van den Driessche (1995).

Bapat, Olesky and van den Driessche (1995) consider the contribution of Fujimoto (1977) but without the assumption $B \geq [0]$. They observe that the results of Fujimoto hold under the following conditions:

$(C1)$ $A \geq [0]$;

$(C2)$ A is irreducible;

$(C3)$ There exists a vector $q \geq [0]$ such that $Bq > Aq$;

$(C4)$ For all $i \neq j$ we have $b_{ij} \leqq a_{ij}$.

These authors give a more direct proof of the results of Fujimoto, by imposing conditions weaker than $(C1) - (C4)$.

They first prove the following lemma.

Lemma 4. Let (A, B) be square matrices of order n satisfying the properties $(C1) - (C4)$. Then $(B - A)^{-1}A$ exists, it is semipositive and irreducible.

Then the same authors prove the following result.

Theorem 18. Let (A, B) be square matrices of order n such that $(B - A)$ is nonsingular and such that $(B - A)^{-1}A$ is semipositive and irreducible. Then there exist $\lambda \in (0, 1)$ and a vector $x > [0]$ such that $Ax = \lambda Bx$. Furthermore, if $Av = \lambda_1 Bv$ and $v \geq [0]$, then $\lambda_1 = \lambda$ and $v = \alpha x$ for some $\alpha > 0$.

Hence, on the ground of Lemma 4, we have that if the pair (A, B) satisfies the properties $(C1) - (C4)$, the thesis of Theorem 18 holds true. Moreover, if (A, B) satisfies $(C1) - (C4)$, then so do the matrices $A^\top, B^\top$ and therefore there exists $\bar{\lambda} > 0$ and a vector $y > [0]$ such that

$$y^\top A = \bar{\lambda} y^\top B.$$

These authors prove also that, by denoting $\rho(A, B)$ the number λ which appears in Theorem 18, we have

$$\rho(A, B) = \frac{\rho((B - A)^{-1}A)}{1 + \rho((B - A)^{-1}A)},$$

i. e.. being $(B - A)^{-1}A$ semipositive,

$$\rho(A, B) = \frac{\lambda^*((B - A)^{-1}A)}{1 + \lambda^*((B - A)^{-1}A)}.$$

The results of Bapat and others (1995) have been reconsidered by Mehrmann, Olesky, Phan and van den Driessche (1999) who are concerned with the relationships betwen the assumptions of Bapat and others (1995), i. e.

$$(B - A) \text{ is nonsingular and } (B - A)^{-1}A \geq [0] \text{ and irreducible,}$$

and one of the basic assumptions of Mangasarian (1971), i. e.

$$y^\top B \geqq [0] \Longrightarrow y^\top A \geqq [0]\,,$$

equivalent to the existence of a square nonnegative matrix H such that

$$A = BH,$$

and hence, if B is nonsingular, such that

$$H = B^{-1}A \geqq [0]\,.$$

Mehrmann and others (1999), similarly to Fujimoto (1977), first note that neither of these two extensions of the Perron-Frobenius theorem is a generalization of the other, as shown by the following examples.

Example 1. If

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then Mangasarian's condition is satisfied, however

$$(B-A)^{-1}A = \begin{bmatrix} -1 & -1 \\ -\frac{1}{2} & -1 \end{bmatrix}.$$

Example 2. If

$$A = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} \end{bmatrix}, \quad B = I,$$

then

$$(B-A)^{-1}A = \begin{bmatrix} 0.037 & 0.5926 \\ 0.5926 & 0.4815 \end{bmatrix},$$

however Mangasarian's condition is not valid for $y \geqq [0]$ and $2y_2 < y_1$.

These authors are concerned with the relationships between the two said approaches. Write $(B-A) = B(I - B^{-1}A)$ (obviously under the assumption that B is nonsingular); then if $(B-A)$ is nonsingular, we have

$$(B-A)^{-1} = (I - B^{-1}A)^{-1}B^{-1}$$

and hence

$$(B-A)^{-1}A = (I - B^{-1}A)^{-1}B^{-1}A.$$

Putting $Z \equiv B^{-1}A \geqq [0]$, the relationship between the two generalizations should become apparent when the equivalence

$$(I-Z)^{-1}Z \geqq [0] \iff Z \geqq [0]$$

holds. A quite immediate result is that if $\lambda^*(Z) < 1$, then $Z \geqq [0]$ implies that $(I-Z)^{-1}Z \geqq [0]$. Indeed, as Z is a nonnegative matrix, with $\lambda^*(Z) < 1$, the matrix $(I-Z)$ is an $\mathcal{M}$-matrix. Thus $(I-Z)^{-1} \geq [0]$ and hence it is nonnegative the product $(I-Z)^{-1}Z$. On the other hand, we have previously observed that if $Z = B^{-1}A \geqq [0]$, the original problem is brought back to the problem

$$\begin{cases} B^{-1}Ax = \lambda x \\ x \geq [0]\,, \quad \lambda \geqq 0 \end{cases},$$

i. e. we have a usual Perron-Frobenius problem.

The reverse implication holds under more complicate conditions. We refer the reader to the paper of Mehrmann and others (1999).

Remark 2. Another interesting result on generalized Perron-Frobenius theorem is given by Peris (1991). Let us consider a *positive splitting* of a square matrix M, i. e. $M = B - A$, with $A \geqq [0]$, $B \geqq [0]$. This author proves the following result.

• Given a square nonsingular matrix M, the following conditions are equivalent:

(a) $M^{-1} \geq [0]$, i. e. $M = (B - A)$ is all-productive.

(b) For all positive splittings of M :

$$M = B - A,\ B \geqq [0]\,,\ A \geqq [0]\,,$$

there exist $v \geq [0]$, $\mu^* \in [0, 1)$ such that

$$Av = \mu^* Bv.$$

However, notice that we cannot ensure the inverse positiveness of M when condition (b) is satisfied only for *some* positive splittings of M. We prove only the implication $(a) \Longrightarrow (b)$. For the converse implication, see Peris (1991). Let $M = B-A$ be a positive splitting of the matrix M and let be $M^{-1} = (B-A)^{-1} \geq [0]$. Then we have $M^{-1}B \geqq [0]$. So, by the Perron-Frobenius theorem, there exist $v \geq [0]$, $\alpha \in \mathbb{R}_+$ such that

$$(M^{-1}B)v = \alpha v. \tag{10}$$

As $(M^{-1}B) = I + M^{-1}A \geqq I$, we know that $\alpha \geqq 1$. Then, premultiplying (10) by M, we obtain

$$Bv = \alpha Mv = \alpha(Bv - Av),$$

that is

$$Av = \left(1 - \frac{1}{\alpha}\right) Bv.$$

Take

$$\mu^* = 1 - \frac{1}{\alpha}.$$

Then $Av = \mu^* Bv$, and $\mu^* \in [0, 1)$. □

D) The main results of Bidard (1996, 2004).

C. Bidard has been concerned with generalizations of the Perron-Frobenius theorem in several papers (some results of Bidard have already been anticipated in Section 3 of the present paper). In Bidard and Zerner (1990, 1991) there are various results on this topic, but these papers are highly technical and cannot be understood by a non mathematical reader. Also the paper of Bidard (1984) treats the problem with a quite general approach. On the contrary, Bidard (1996, 2004) takes into consideration an extension of the Perron-Frobenius theorem directly applicable to linear joint production models. The main results of these last two works are the following ones. First we point out that it is not not assumed that in the pair of square matrices (A, B), we have $A \geqq [0]$, but it is assumed that, for every $i = 1, ..., n$, it holds $B_i \geq [0]$ or that, for every $j = 1, ..., n$, it holds $B^j \geq [0]$. A value λ (not necessarily positive, but real) such that $\det(\lambda B - A) = 0$, is called a *generalized eigenvalue of* (A, B) or, simply, an *eigenvalue* of (A, B). "Basic single productions" means that the matrix B has exactly one positive element in every row and every column and that the matrix $(\alpha B + A)$ is semipositive and indecomposable for α great enough. The symbol $\exists!$ means "existence and uniqueness up to a factor". Note that if A is not positive and $B = I$, the properties obtained by Bidard consitute a first extension of the classical Perron-Frobenius theorem for the pair (A, I). In fact we have in this case a Metzlerian

Bidard adopts the following notation

$$S \equiv \left\{ \lambda : (\lambda B - A)^{-1} > [0] \right\}.$$

A first result (previously given when introducing the definition of all-engaging systems and of g-all-engaging systems is:

$$\lambda \in S \iff \left\{ \begin{array}{c} \exists y^0 \geq [0] \,, \ (\lambda B - A)y^0 \geq [0] \,, \\ \{y \geq [0] \,, \ (\lambda B - A)y \geq [0]\} \Longrightarrow y > [0] \end{array} \right\}.$$

The set S may be empty, however, in the converse case Bidard (1996, 2004) proves the following result.

Theorem 19. If $S \neq \varnothing$, then this property is equivalent to property (i) or (ii) below:

(i)

$$\begin{aligned} \exists \Lambda, \exists! q &> [0] \,, Aq = \Lambda B q \\ \nexists y &\geq [0] \,, \ (\Lambda B - A)y \geq [0] \,. \end{aligned}$$

(ii)

$$\begin{aligned} \exists! q &> [0] \,, Aq = \Lambda B q \\ \exists \pi &> [0] \,, \ \pi^\top A = \Lambda \pi^\top B. \end{aligned}$$

(The generalized eigenvalue Λ is not restricted to be positive, as well as the scalar λ).

Furthermore, the following properties hold.
(*iii*)

$$q = \lim_{\lambda \longrightarrow \Lambda^+} (\lambda B - A)^{-1} d \big/ \left\| (\lambda B - A)^{-1} d \right\| > [0] \text{ for any } d \geq [0] \,.$$

(*iv*) Λ is a simple root of $\det(\lambda B - A) = 0$ and the unique eigenvalue of (A, B) to which it corresponds a semipositive eigenvector.

(*v*) $S = (\Lambda, \lambda_0)$, λ_0 finite, except for "basic single productions".

(*vi*) For any generalized eigenvalue λ_h of (a, B) we have

$$|\lambda_0 - \lambda_h| \geqq \lambda_0 - \Lambda \quad (\text{"circle rule"}).$$

Note that if it is assumed that also A is nonnegative in the pair (A, B), it is possible to show that in the previous relations we have $\lambda > 0$ and that the system (A, B) is λ-all-engaging in an interval $S' = (\lambda_{\min}, \lambda_{\max})$. See Bidard (1996, 2004).

Then Bidard (1996, 2004) is concerned with λ-all-productive systems and defines the set

$$T \equiv \left\{ \lambda : (\lambda B - A)^{-1} \geq [0] \right\}.$$

The assumptions are the same of the λ-all-engaging case: $B_i \geq [0]$ for all i or $B^j \geq [0]$ for all j, however in "basic simple production" the indecomposability condition of $\alpha B + A$ is not required.

We recall that, when $T \neq \varnothing$:

$$\lambda \in T \iff \left\{ \begin{array}{c} \exists y^0 \geq [0] \,, \; (\lambda B - A) y^0 > [0] \,; \\ (y \geq [0] \,, \; (\lambda B - A) y > [0]) \Longrightarrow y > [0] \end{array} \right\}.$$

Theorem 20. If T is nonempty, the following properties hold.
(*i*) $\exists \Lambda$, $\exists q \geq [0] : Aq = \Lambda B q$,
$\nexists y \geq [0] : (\Lambda B - A) y > [0]$.
(*ii*) $\exists \pi \geq [0] : \pi^\top A = \Lambda \pi^\top B$.
(*iii*) $T = (\Lambda, \lambda_0]$, λ_0 finite except for "basic simple production".
(*iv*) Circle rule.

Note that also for the λ-all-productive case it is not required $A \geqq [0]$. If $M = B - A$, with $A \geqq [0]$ and $B \geqq [0]$ (case of "positive splitting"), we can apply the previously quoted results of Peris (1991).

Remark 3. Our review of the Perron-Frobenius properties related to the pair (A, B) is obviously not exhaustive. Besides the papers previously quoted, we point out also the papers of Mehrmann, Nabben and Virnik (2008), Erdelyi (1967), Kershaw (1972/73), Drandakis (1966), Thompson and Weil (1970, 1972). Finally (see Kershaw (1972/73)), we observe that if the square matrices A, B commute ($AB = BA$), then they share the same eigenvectors. Obviously, this assumption is of scarse or zero economic interest.

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