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Some Basic Mathematical Properties on the von Neumann Growth Model: A Didactic Note

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Abstract. We give a survey of the main mathematical properties concerning the classical von Neumann growth model, properties often scattered in several papers and books and not always proved in a fully correct way. Hence, the present note may have some didactic utility.

Key Words: von Neumann growth model, von Neumann equilibrium solutions.

1. Introduction

The von Neumann economic growth model (von Neumann (1945-46)) is perhaps one of the most investigated models in economic growth theory and in mathematical economics in general. However, we think that some basic mathematical results related to the said model are scattered in various papers and books and that someone of the said results and proofs are not quite completely correct. The aim of the present paper is to give a unified didactic treatment of the basic properties of the classical von Neumann model, avoiding the use of the Theory of Games or of other too specific tools of Mathematics. The paper is organized as follows. Section 2 treats the problem of the existence of solutions; Section 3 is concerned with the equality between the maximal growth factor α^* and the minimal interest factor β^*, together with some considerations on the irreducibility of the pair of matrices (A, B). Section 4 presents a nonlinear generalization of the von Neumann model, essentially due to Gale (1956) and Karlin (1959). Section 5 contains some additional remarks on the classical von Neumann model and the final Section 6 presents some conclusions.

All numbers, matrices and vectors considered are real. A and B are matrices of order (m, n). The i-th row of A is denoted by A_i, $i = 1, ..., m$, and the j-th column of A is denoted by A^j, $j = 1, ..., n$. Obviously, the same holds for B. The notation $[0]$ is used to denote a zero vector or a zero matrix, while the notations $A > [0]$, $A \geqq [0]$, $A \geq [0]$ are used, respectively, for a *positive matrix,* i. e. $a_{ij} > 0, \forall i, j$; a *nonnegative matrix, i. e.* $a_{ij} \geqq 0$, $\forall i, j$; a *semipositive matrix,*

i. e. $A \geqq [0]$, $A \neq [0]$. $A < [0]$, $A \leqq [0]$, $A \leq [0]$ are defined in a similar way. The same notations apply to a vector $x \in \mathbb{R}^n$ with respect to the zero vector $[0] \in \mathbb{R}^n$. Vector of $\mathbb{R}^n$ are considered as *column vectors,* and hence *row vectors* are the transpose of column vectors: $x^\top$. $A^\top$ is the transpose of the matrix A; $e \in \mathbb{R}^n$ is the vector with all components equal to one:

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The notation e^i is used to denote the i-th *standard vector of* $\mathbb{R}^n$, i. e. that vector containing all zeros, except for the i-th component, which is equal to one. The *support* of a nonnegative vector $x \in \mathbb{R}^n$ is defined as follows:

$$\text{supp}(x) = \{i : x_i > 0\}.$$

Let us consider a linear production and exchange model implying n *processes* or *industries* or *activities* or *sectors* and m *goods,* model described by a pair (A, B) of (m, n) nonnegative matrices. The j-th process (or activity) uses the inputs described by A^j, the j-th column of the input matrix A. So A is called the *input matrix* and a_{ij} denotes the amount of good i consumed by activity j. The j-th process obtains the outputs described by the corresponding j-th column B^j of the matrix B. So b_{ij} represents the amount of good i produced by activity j. Each process is supposed to be activated at an intensity level equal to one; x, $y^\top$, α, β, describe, respectively, the column vector of activity intensities, the row vector of unit prices, the uniform growth factor (1+ growth rate) the uniform interest factor (1+ interest rate). In all what follows it is supposed that A and B (both of order (m, n)) are nonnegative:

$$A \geqq [0], \quad B \geqq [0]. \tag{1}$$

We stress again that in matrices A and B their rows correspond to the goods (respectively, the means of production and the products) and their columns correspond to the activities of the economic system. Hence, a column of A and thye corresponding column of B represent a production process activated at an intensity level equal to one.

The pair (A, B) is called a *von Neumann technology* if, moreover, verifies the following two properties, inposed by Kemeny, Morgenstern and Thompson (1956):

$$A^j \geq [0], \quad \forall j = 1, ..., n. \tag{2}$$

$$B_i \geq [0], \quad \forall i = 1, ..., m. \tag{3}$$

Assumption (2) says that every process needs at least one good as input; assumption (3) says that every good is an output of some process.

Then the classical von Neumann model, in the version proposed by Kemeny, Morgenstern and Thompson (1956), can be described by the following relations

$$(B - \alpha A)x \geqq [0]\,, \quad x \geq [0]\,, \quad \alpha > 0 \tag{4}$$

$$y^\top (B - \beta A) \leqq [0]\,, \quad y \geq [0]\,, \quad \beta > 0 \tag{5}$$

$$y^\top (B - \alpha A)x = 0 \tag{6}$$

$$y^\top (B - \beta A)x = 0 \tag{7}$$

$$y^\top Bx > 0. \tag{8}$$

Relation (4) ensures that gross productions Bx are able to cover the inter-industry requirements for the industrial consumption Ax with a *balanced growth* described by α, i. e. $x_{t+1} = \alpha x_t$, $t = 1, 2, ...$, when the inputs for one period are supplied from the outputs of the preceeding period. The *maximum growth factor,* denoted by α^*, is the maximum value of α such that an intensity vector $x \geq [0]$ exists, i. e.

$$\alpha^* = \max\{\alpha \in \mathbb{R}_+ : \exists x \geq [0]\,,\ Bx \geqq \alpha Ax\}\,.$$

Relation (5) ensures that at prices described by y no activity gains extra-profits, beyond the ones corresponding to the interest factor. In other words, (5) is a necessary condition for competitive equilibrium at β (as under constant returns to scale no process can in equilibrium give positive pure profits). The *minimum interest factor,* denoted by β^*, is the minimum value of β such that a price vector exists, i. e.

$$\beta^* = \min\left\{\beta \in \mathbb{R}_+ : \exists y \geq [0]\,,\ y^\top B \leqq \beta y^\top A\right\}.$$

Relation (6) is the so-called *rule of free goods,* i. e. it makes equal to zero the price of each over-produced good. Relation (7) is the so-called *rule of unprofitable processes* or *inferior activities rule,* as it implies that every process obtaining a sector interest factor inferior to the equilibrium factor β is not activated. Relation (8) imposes that the total value of the output is positive.

We have to note that assumptions (2), (3) and (8) of Kemeny, Morgenstern and Thompson substitute the original assumption

$$A + B > [0]$$

of von Neumann, less realistic from an economic point of view, but somewhat reasonable if assumed together with (2) and (3).

Remark 1. Assumption (2), we shall call also $(KMT)_1$, i. e. every activity has some good as input, can be equivalently expressed as

$$x \geq [0] \Longrightarrow Ax \geq [0]$$

or as

$$y > [0] \Longrightarrow y^\top A > [0]\,.$$

Assumption (3), we shall call also $(KMT)_2$, i. e. every good is an output of some activity, can be equivalently expressed as

$$y \geq [0] \Longrightarrow y^\top B \geq [0]$$

or as

$$x > [0] \Longrightarrow Bx > [0]\,.$$

See, e. g., Abraham-Frois and Berrebi (1979), Robinson (1973). The von Neumann model satisfying $(KMT)_1$ and $(KMT)_2$ is called "normal" by Łos (1971).

Definition 1. A quadruplet (x, y, α, β) satisfying relations (4)-(8) is called an *equilibrium solution* of the von Neumann model.

We have the following simple but basic result.

Lemma 1. If (x, y, α, β) is an equilibrium solution of the von Neumann model, then $\alpha = \beta > 0$.

Dim. From the complementarity conditions (6) and (7) we have

$$\beta y^\top Ax = \alpha y^\top Ax = y^\top Bx > 0.$$

Hence $y^\top Ax > 0$ and $\alpha = \beta > 0$. $\square$

Therefore from Lemma 1 we obtain the following "simplified" definition of equilibrium solutions for a von Neumann technology described by relations (4)-(8).

Definition 2. A triplet (x, y, α), with $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^m_+$, $\alpha \in \mathbb{R}_+$, is an equilibrium solution of the von Neumann technology if the following relations hold

$$\alpha Ax \leqq Bx, \quad x \geq [0] \tag{9}$$

$$\alpha y^\top A \geqq y^\top B, \quad y \geq [0] \tag{10}$$

$$y^\top Bx > 0. \tag{11}$$

Łos (1971) calls the number $\alpha > 0$ appearing in Definition 2 "equilibrium level", whereas Kemeny, Morgenstern and Thompson (1956) call the same quantity "allowable". Therefore, if (8) (subsequently (11)) is satisfied, the complementarity relations (6) and (7) can be deleted from the model. Indeed, the complementarity conditions (6) and (7) follow straightforwardly from (9) and (10). Furthermore, it must be noted that:

• There may exist several distinct equilibrium levels satisfying the von Neumann model; this number is however finite and does not exceed $\min(m, n)$. See Kemeny, Morgenstern and Thompson (1956), Morgenstern and Thompson (1976), Murata (1977). Also Roemer (1980) gives a proof of this property, however in a quite complex way. Intuitively: each good is made in at most one equilibrium and each activity is used in at most one equilibrium, so there are at most $\min(m, n)$ equilibria. Mc Kenzie (1967) gives a quite sghort proof of the said property by means of the following results (result (I) will be proved in the next section, result (II) is due to Thompson (1956)).

(I) Under $(KMT)_1$ and $(KMT)_2$ there exists a von Neumann equilibrium (x, y, α) for the model (A, B). In particular, if α is maximal for (9) or minimal for (10), there is an equilibrium for α.

(II) If (x, y, α) and (x', y', α') are von Neumann equilibria, where $\alpha \neq \alpha'$, then $y_i b_{ij} x_j > 0$ and $y'_h b_{hk} x'_k > 0$ imply $i \neq h$ and $j \neq k$.

It follows directly fom (I) and (II) that the number of equilibria for a model (A, B) satisfying $(KMT)_1$ and $(KMT)_2$ is at least one and at most $\min(m, n)$.

• The values $\alpha = \alpha^* = \beta^*$ are parts of the solutions of the above von Neumann model (property (I) above. See the next section).

2. The existence of solutions

As it is known, the original technique used by von Neumann to prove the existence of solutions of his model (where condition (11) does not appear, nor appear assumptions (2) and (3), but it appears the assumption $(A + B) > [0]$) is based on a generalization of Brouwer's fixed point theorem and on saddle points properties of the function $y^\top Bx/y^\top Ax$. These techniques are in effect rather involved and inadequate to treat open versions of the model and/or improvements of the assumption $(A + B) > [0]$. The approach of Kemeny, Morgenstern and Thompson (1956) may be seen as an attempt to amend some of the criticisms raised by the assumption $(A+B) > [0]$. These authors introduce $(KMT)_1$ and $(KMT)_2$ and condition (11), but use in their proof tools of the Theory of Games, not always available by students of Economic Theory. The proof of Howe (1960) does not use the Theory of Games, but it is performed by means of a theorem of Tucker (1956) on linear inequality systems, a theorem of a "specialistic" nature. The same is done by Nikaido (1968) and by Murata (1977), who, however, offers a rather comprehensive treatment of the von Neumann classical model. The contributions of Gale (1960) are various and important, as this author proves that under $(KMT)_1$ and $(KMT)_2$, the quantities α^* and β^* exist (finite and positive), the problem

$$Q(\alpha): \qquad \max_{\alpha, x} \{\alpha \in \mathbb{R}_+ : (B - \alpha A)x \geqq [0], \quad x \geq [0]\}$$

admits solutions, with $\alpha \in (-\infty, \alpha^*]$, and the problem

$$P(\beta): \qquad \min_{\beta, y} \left\{\beta \in \mathbb{R}_+ : y^\top (B - \beta A) \leqq [0], \quad y \geq [0]\right\}$$

admits solutions, with $\beta \in [\beta^*, +\infty)$.

Furthermore, Gale introduced the concept of *irreducibility* for the pair (A, B) (see further), in order to obtain the equality $\alpha^* = \beta^*$. However, the proof of Gale (1960) is not satisfactory, as this author states, in a footnote, that "a standard *compactness* argument is needed here to show that there actually exists a semipositive x^0 such that $(B - \alpha_0 A)x^0 \geqq [0]$". Unfortunately the related function of problem $Q(\alpha)$ is *not* continuous, as pointed out by Glycopantis (1970), with reference to a more general non-polyhedral von Neumann model, previously considered by Gale (1956). Indeed, this function is *upper semicontinuous,* which is enough for our purposes: see, e. g., Crouzeix (1981), but this fact should be expressively stated. The subsequent note of Gale (1972) does not solve the question, as this note proves the existence of an "equilibrium level" satisfying (9)-(11). The treatment of Łos (1971) is more complete, but done with reference to general spaces, and hence to general von Neumann models. We begin by giving a short proof of the existence of the maximal growth rate α^* (the proof of the existence of β^* is similar).

Lemma 2. Assume that $(KMT)_1$ and $(KMT)_2$ are satisfied. Then the number α^* solution of $Q(\alpha)$ exists finite and positive. Hence for some vectors $x^* \geq [0]$ it holds $\alpha^* A x^* \leqq B x^*$.

Proof. Let us denote by S the set

$$S = \{\alpha \,|\, \alpha \in \mathbb{R}, \text{ there exists } x \geq [0] \text{ such that } \alpha A x \leqq B x\}.$$

First we prove that the set S contains positive elements. Let be $x > [0]$. From $(KMT)_2$ it results $Bx > [0]$. Therefore for $\alpha > 0$ sufficiently small the inequality $\alpha A x \leqq Bx$ is satisfied, so that $\alpha \in S$. It remains to show that S is bounded from above. From assumption $(KMT)_1$, for α sufficiently large, the sum of the elements of each column of the matrix $B - \alpha A$ is negative. From this, denoting by e the *sum vector,* i. e. the vector of $\mathbb{R}^n$ whose components are all equal to one, we obtain $e^\top (B - \alpha A) < [0]$ for all $x \geq [0]$. Therefore, for α sufficiently large there does not exist a semipositive vector $x \geq [0]$ such that $(B - \alpha A)x \geqq [0]$. Hence the number $\alpha^* = \sup\{S\}$ exists finite and positive.

Now consider the series $\{\alpha_k\}_{k=1}^{\infty} \subset S$ such that $\alpha_k \longrightarrow \alpha^*$ when $k \longrightarrow +\infty$. From the definition of S to each number α_k it corresponds a semipositive vector w^k such that

$$\alpha_k A w^k \leqq B w^k, \quad k = 1, 2, ... \tag{12}$$

Let us denote $x^k = w^k / \left\| w^k \right\|$. By dividing both members of inequality (12) by $\left\| w^k \right\|$, we obtain

$$\alpha_k A x^k \leqq B x^k, \quad k = 1, 2, ... \tag{13}$$

The bounded series $\left\{ x^k \right\}_{k=1}^{\infty}$ has a limit point. Let x^* be the said limit point. We have $x^* \geq [0]$ (more precisely we have $\|x^*\| = 1$) and from (13) we obtain

$$\alpha^* A x^* \leqq B x^*. \quad \square$$

We have already said that the number

$$\alpha^* = \max\{\alpha \mid \alpha \in \mathbb{R}_+,\ \text{there exists } x \geq [0] \text{ such that } \alpha Ax \leqq Bx\}$$

is called *growth factor* or *expansion factor* of the von Neumann technology. Any vector $x^* \geq [0]$ such that $\alpha^* Ax^* \leqq Bx^*$ is called *vector of optimal intensity.*

In what follows we present the proof of the existence of equilibrium given by Łos (1971). First we need the following theorem of the alternative (see also Gale (1972)).

Lemma 3. Let M be a matrix of order (m, n) and b a vector of $\mathbb{R}^n$. Then the system

$$Mz \geqq b, \quad z \geqq [0]$$

has a solution if and only if the system

$$y^\top M \leqq [0]\,,\ y \geqq [0]\,,\ y^\top b > 0$$

has no solution.

Proof. The above lemma is nothing but a formal modification of the classical Farkas-Minkowski theorem of the alternative, i. e. the system $Ax = b$, $x \geqq [0]$ has a solution if and only if the system $y^\top A \leqq [0]\,,\ y^\top b > 0$ has no solution. Indeed, consider the Farkas-Minkowski theorem, the matrix A and the vector x partitioned as follows:

$$A = [M \mid -I]\,; \quad x = \begin{bmatrix} z \\ v \end{bmatrix}.$$

Then, apply to the pair (A, x) the Farkas-Minkowski theorem of the alternative in order to obtain at once Lemma 3. □

An immediate consequence of Lemma 3 is the following statement, due to Łos (1971).

Lemma 4. Let a vector $\hat{x} \geq [0]$ and a number $\alpha > 0$ satisfy the relation

$$(B - \alpha A)\hat{x} \geqq [0]\,.$$

A necessary and sufficient condition for the existence of a vector $\hat{y} \geq [0]$ such that

$$\begin{aligned} y^\top(B - \alpha A) &\leqq [0] \\ \hat{y}B\hat{x} &> 0 \end{aligned}$$

(i. e. the triplet $\hat{x}, \hat{y}, \alpha$ form an equilibrium solution of the "reduced" von Neimann model) is that the inequality

$$B\hat{x} \leqq (B - \alpha A)x, \quad x \geqq [0]\,,$$

admits no solution.

Proof. Put in Lemma 3, $M = (B - \alpha A)$, $b = B\hat{x}$. $\square$

In order to perform the proof of the existence of equilibria in a "reduced" von Neumann model we need the notion of *support* of a nonnegative vector. We recall this notion, already given in Section 1.

Definition 3. Let be given a vector $x \geqq [0]$, $x \in \mathbb{R}^n$; then the *support* of x is defined as follows:

$$\operatorname{supp}(x) = \{i : x_i > 0\}.$$

Lemma 5. Let $x \geqq [0]$ and $z \geqq [0]$ be two vectors of the same order. A necessary and sufficient condition for the existence of a number $\gamma > 0$ such that $x \leqq \gamma z$, is that

$$\operatorname{supp}(x) \subset \operatorname{supp}(z).$$

Proof. Evident.

Finally, we give the "existence proof".

Theorem 1. If matrices A and B satisfy the assumptions $(KMT)_1$ and $(KMT)_2$, then there exists a vector

$$x^* \in V^* = \{x^* : x^* \geq [0], \text{ such that } \alpha^* A x^* \leqq B x^*\}$$

and a vector $y^* \in \mathbb{R}^m_+$ such that the triplet $(x^* y^*, \alpha^*)$ is an equilibrium solution of the von Neumann model.

Proof. For every vector $x^* \in V$ the inequality $\alpha^* A x^* \leqq B x^*$ is satisfied. By Lemma 4 we have to prove that there exists a vector $x^* \in V$ such that the inequality

$$Bx^* \leqq (B - \alpha^* A)x, \quad x \geqq [0]$$

admits no solution. Let us assume that this assertion is not true: for every vector $x^* \in V^*$ there exists a vector $x \geqq [0]$ such that

$$Bx^* \leqq (B - \alpha^* A)x.$$

From Lemma 5 it results that for every vector $x^* \in V$ there exists a vector $x \geqq [0]$ such that

$$\operatorname{supp}(Bx^*) \subset \operatorname{supp}(B - \alpha^* A)x.$$

We now build a vector $\tilde{x} \in V^*$ with the property

$$\operatorname{supp}(B - \alpha^* A)x \subset \operatorname{supp}(B - \alpha^* A)\tilde{x}, \quad \forall x \in V^*. \tag{14}$$

For this purpose we choose in V^* the vectors x^j, $j = 1, ..., m$, with the following rule: if there exists a vector $x \in V^*$ such that the j-th component of $(B - \alpha^* A)x$ is positive, then $v^j = x$; otherwise $x^j = 0$. We now put

$$\tilde{x} = \frac{1}{m}(x^1 + x^2 + ... + x^m).$$

Evidently $\tilde{x} \in V^*$ whereas inclusion (14) holds true by construction of the element $\tilde{x}$. As $\tilde{x} \in V^*$, there exists a vector $x \geqq [0]$ such that

$$B\tilde{x} \leqq (B - \alpha^* A)x.$$

From this inequality and from the inequality $B\tilde{x} \geqq [0]$ it results $x \in V^*$. Similarly we have

$$\text{supp}(B\tilde{x}) \subset \text{supp}(B - \alpha^* A)x.$$

By (14) we find

$$\text{supp}(B\tilde{x}) \subset \text{supp}(B - \alpha^* A)\tilde{x}.$$

Applying Lemma 5 we deduce the existence of a number $\gamma > 0$ such that

$$B\tilde{x} \leqq \gamma(B - \alpha^* A)\tilde{x}. \tag{15}$$

This inequality remains true if we substitute γ with any number $\tilde{\gamma} > \gamma$, so that we can assume $\gamma > 1$. From (15) we obtain

$$\gamma\alpha^* A\tilde{x} \leqq (\gamma - 1)B\tilde{x},$$

that is

$$\frac{\gamma}{\gamma - 1}\alpha^* A\tilde{x} \leqq B\tilde{x}. \tag{16}$$

But $\frac{\gamma}{\gamma-1}\alpha^* > \alpha^*$, which, together with relation (16) contradicts the definition of α^* as the maximal technological expansion factor. $\square$

Remark 2. An alternative way to prove Lemma 4 is suggested by Łos (1971), who takes into consideration the following pair of linear programming problems (primal and dual):

$$(P): \qquad \begin{cases} \max y^\top(B\hat{x}) \\ y^\top(B - \alpha A) \leqq [0] \\ y^\top e \leqq 1, \quad y \geqq [0], \end{cases}$$

where $e^\top = [1, 1, ...1]$. The value of this problem is negative, since $y = [0]$ satisfies the constraints. It is also finite, because the set of feasible vectors is compact, The value is positive if and only if $\hat{x}$ is in equilibrium with some $\hat{y}$ at level α. The dual of this problem is

$$(D): \qquad \begin{cases} \min \rho \\ (B - \alpha A)x + \rho e \geqq B\hat{x} \\ x \geqq [0], \ \rho \geqq 0. \end{cases}$$

The value of both problems is the same, thus if the value is positive, then $\hat{x}$ is in equilibrium at the level α, if the value is zero, then $B\hat{x} \leqq (B - \alpha A)x$, $x \geqq [0]$ has a solution.

Remark 3. Another way to prove Lemma 2 is suggested again by Łos (1971). Assume that $(KMT)_1$ and $(KMT)_2$ hold true. Let $q > [0]$ be a given vector of $\mathbb{R}^n$ and consider the set $Z(\alpha) = \left\{x \geqq [0] : Bx \geqq \alpha Ax,\ q^\top x = 1\right\}$. We have that α^* is the number

$$\alpha^* = \max\{\alpha : Z(\alpha) \neq \varnothing\}.$$

Then we have the following result:
- α^* exists and it holds

$$0 < \alpha^* < +\infty.$$

Indeed, from $(KMT)_2$ we get the existence of a positive (sufficiently small) number α, such that $Z(\alpha) \neq \varnothing$. Furthermore, since $Z(\alpha)$ is compact and $Z(\alpha_1) \subset Z(\alpha_2)$ for $\alpha_2 \leqq \alpha_1$, we see that for $\bar{\alpha} = \sup\{\alpha : Z(\alpha) \neq \varnothing\}$, the set $Z(\bar{\alpha})$ is the intersection of all $Z(\alpha)$ with $\alpha < \bar{\alpha}$ and thus must be nonempty. It follows that $\bar{\alpha} < +\infty$, for in the contrary case we would have $\alpha Ax \leqq Bx$ for any x in $Z(\bar{\alpha})$ and $\alpha > 0$, which implies $Ax = [0]$, in contradiction with $(KMT)_1$.

Remark 4. Similarly to what proved in Lemma 2, we get the existence, under the assumptions $(KMT)_1$ and $(KMT)_2$, of the minimal interest factor β^*, i. e. of a number $\beta^* > 0$ solution of

$$P(\beta): \qquad \min_{\beta, y}\left\{\beta \in \mathbb{R}_+ : y^\top (B - \beta A) \leqq [0],\quad y \geq [0]\right\}.$$

We do not repeat the proof for the above assertion.

An alternative proof of Theorem 1 can be found in the paper of Gale (1972) where, however, (we repeat) no proof of the existence of a positive maximal expansion factor α^* is given.

For other considerations on mathematical properties of the classical von Neumann model see Abraham-Frois and Berrebi (1979), Afriat (1987a, 1987b), Bruckman and Weber (1971), Fujimoto (1975), Giorgi and Meriggi (1987, 1988), J. and M. W. Łos (1974), Łos (1979), Makarov and Rubinov (1977), Morgenstern and Thompson (1976), Moeschlin (1974), Morishima (1964, 1969), Murata (1977), Nikaido (1968, 1970), Woods (1978).

3. Equality between the maximal growth factor and the minimal interest factor

We have seen that under $(KMT)_1$ and $(KMT)_2$ α^* and β^* both exist positive and are two equilibrium levels. Kemeny, Morgenstern and Thompson

(1956) and Morgenstern and Thompson (1976) speak of "allowable levels". They prove that, always under $(KMT)_1$ and $(KMT)_2$ there are at most $\min(m, n)$ allowable levels. A basic result of Gale (1960) is that, under $(KMT)_1$ and $(KMT)_2$ it holds $\beta^* \leqq \alpha^*$. Gale he gives a counterexmple in which $\beta^* < \alpha^*$. We follow the proof of Gale (1960). See also Abraham-Frois and Berrebi (1979), Giorgi and Meriggi (1987, 1988). We need a previous result. See Gale (1960).

Lemma 6. (Semipositive solutions of homogeneous inequalities). Exactly one of the following alternatives holds: either

$$Ax \leqq [0]$$

has a semipositive solution, or

$$y^\top A > [0]$$

has a nonnegative solution.

Theorem 2. Let $(KMT)_1$ and $(KMT)_2$ be verified; then it holds

$$\beta^* \leqq \alpha^*.$$

Proof. Let $C = B - \alpha^* A$. The inequality $Cx > [0]$ has no nonnegative solution, for if x were such a solution, then we would have $\alpha^* Ax < Bx$, or

$$(\alpha^* + \delta)Ax \leqq Bx$$

for some positive δ, so that α^* would not be maximal. Now apply Lemma 6: we know that there is a vector $y \geq [0]$ such that $y^\top C \leqq [0]$, or equivalently

$$\alpha^* y^\top A \geqq y^\top B.$$

Thus it follows from the definition of β^* that $\alpha^* \geqq \beta^*$. $\square$

As previously said, Gale (1960) gives an example where $\beta^* < \alpha^*$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here we have $\alpha^* = \sqrt[3]{2}$, $\beta^* = 1$.

The original paper of von Neumann imposed a sufficient condition to have $\alpha^* = \beta^*$, i. e. $(A + B) > [0]$. This condition, together with $(KMT)_1$ and $(KMT)_2$, introduced later by Kemeny, Morgenstern and Thompson (1956), is considered by Nikaido (1968), who discusses in Remark 1 (p. 147) its role in obtaining uniqueness between α^* and β^*. Gale (1960) on the other hand, introduces a perhaps more acceptable (from an economic point of view) assumption of *irreducibility* of the pair (A, B). This Gale irreducibility is equivalent to the non existence of a group of *independent goods;* such a group exists, and in this case the von Neumann technology is said to be *reducible* or *Gale-reducible,* if under assumptions $(KMT)_1$ and $(KMT)_2$, a subset of k activities $(1 \leqq k \leqq n)$ and a subset of h goods $(1 \leqq h < m)$ exist, such that each good of this last subset is output of at least an activity of the firts subset, but none of the same activities uses goods not pertaining to the group. More formally:

Definition 4. Given $(KMT)_1$ and $(KMT)_2$, the von Neumann technology is reducible (in the sense of Gale) if we can reorder the columns of A and B and the rows of A and B in such a way that A and B take the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ [0] & A_{22} \end{bmatrix}; \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where the rows of B_{11} are all semipositive. We say that the technology is irreducible if such a decomposition is not possible.

In other words, the model (A, B), where $(KMT)_1$ and $(KMT)_2$ hold, is reducible if there exist subsets $M_1 \neq \varnothing$, $M_2 \neq \varnothing$ of the index set $M = \{1, 2, ..., m\}$ and subsets N_1, N_2 of the index set $N = \{1, 2, ..., n\}$ such that

$$M_1 \cup M_2 = M,\ M_1 \cap M_2 = \varnothing,\ N_1 \cup N_2 = N,\ N_1 \cap N_2 = \varnothing,$$

with $a_{ij} = 0$ for every $i \in M_2$ and $j \in N_1$ and with $b_{ij} > 0$ for each $i \in M_1$ and some $j \in N_1$.

Remark 5. Robinson (1973) calls the above definition "technological reducibility" and gives the following equivalent definition of technological irreducibility and technological reducibility:

• The pair (A, B), where $(KMT)_1$ and $(KMT)_2$ hold, is Gale-irreducible if for every vector $x \in \mathbb{R}^n$, $x \geq [0]$, such that $\text{supp}(Ax) \subset \text{supp}(Bx)$, we have $\text{supp}(Ax) = \{1, ..., m\}$, i. e. $Ax > [0]$.

Therefore the pair (A, B) is Gale-reducible when there exists at least a vector $x \geq [0]$ such that $\text{supp}(Ax) \subset \text{supp}(Bx)$, with $Ax \not> [0]$.

Robinson (1973) does not prove the equivalence between his definition and Definition 4; this equivalence is proved, for instance, by Abraham-Frois and Berrebi (1979). See also Groth (1986). Robinson (1973) remarks that if $(KMT)_2$ holds for the irreducible pair (A, B), then $(KMT)_1$ holds: indeed, if $(KMT)_1$ does not hold, there would be a vector $x \geq [0]$ such that $Ax = [0]$ and for this vector we would have $\text{supp}(Bx) \supset \text{supp}(Ax) = \varnothing$, contradicting the irreducibility assumption. If A is positive, then the pair (A, B) is technologically irreducible. On the other hand, the multiple production model (I, B), with I identity matrix and B square, is technologically reducible.

Now we prove the basic result of Gale (1960). See also Abraham-Frois and Berrebi (1979), Groth (1986), Giorgi and Meriggi (1987, 1988), Murata (1977). Takayama (1985) is not fully clear on this subject.

Theorem 3. Assume that conditions $(KMT)_1$ and $(KMT)_2$ are verified. If the model (A, B) is Gale-irreducible, then $\alpha^* = \beta^*$.

Proof. Since $\alpha^* \geqq \beta^*$ by Theorem 2, it is sufficient to verify that $\alpha^* \leqq \beta^*$. We know that there exists $x \geq [0]$ such that $\alpha^* Ax \leqq Bx$ and there exists $y \geq [0]$ such that $\beta^* y^\top A \geqq y^\top B$. Therefore

$$\alpha^* y^\top Ax \leqq y^\top Bx \leqq \beta^* y^\top Ax. \tag{17}$$

Let M_1 be $\{i : B_i x > 0\}$. By taking N_1 as $\{j : x_j > 0\}$, we must have $a_{ij} = 0$ for every $i \in M_2$ and $j \in N_1$, since otherwise we would get some $i \in M_2$ for which $A_i x > 0$ and $B_i x = 0$, and hence we could not have $\alpha^* A_i x \leqq B_i x$ for all $i \in M$. Now, from the irreducibility assumption, $M_2 = \varnothing$ follows, whence $Bx > [0]$. Since $y \geq [0]$, $y^\top Bx > 0$, implying $y^\top Ax > 0$ and hence $\alpha^* \leqq \beta^*$ in view of (17). □

Robinson (1973) introduces another definition of irreducibility for the model (A, B), called "economic irreducibility" and which may be considered a "dual" version of Gale's definition.

Definition 5. The pair (A, B), where $(KMT)_1$ and $(KMT)_2$ hold is economically irreducible if the pair $(B^\top, A^\top)$ is (Gale)-technologically irreducible. In other words: for each vector $y \geq [0]$ such that $\text{supp}(y^\top B) \subset \text{supp}(y^\top A)$ we have $y^\top B > [0]$, i. e. $\text{supp}(y^\top B) = \{1, ..., n\}$. Therefore the pair (A, B) is (Gale)-economically reducible if there exists a vector $y \geq [0]$ such that $\text{supp}(y^\top B) \subset \text{supp}(y^\top A)$, with $y^\top B \not> [0]$.

We can also say that (A, B) is economically reducible if, by suitable permutations of the rows and columns of A and B (not necessarily the same) it results

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad B = \begin{bmatrix} B_{11} & B_{12} \\ [0] & B_{22} \end{bmatrix},$$

with $A_{21} \geq [0]$.

Robinson (1973) remarks that if the model (A, B) is economically irreducible, the maximal expansion factor α^* and the minimal interest factor β^* are equal: hence economic irreducibility is another sufficient condition to have $\alpha^* = \beta^*$. Moreover, technological irreducibility and economic irreducibility are two independent properties, in the sense that a von Neumann technology may satisfy one property but not the other one. Abraham-Frois and Berrebi (1979) give some numerical examples to show this assertion. See also Giorgi (2007). When B is positive, (A, B) is economically irreducible. If A is square and if the identity matrix is represente by I, then the technological irreducibility of (A, I)

is equivalent to the irreducibility of A in the usual sense. Similarly, economic irreducibility of (I, B) is equivalent to the irreducibility of B in the usual sense. Robinson (1973) remarks that if $(KMT)_1$ holds for an economically irreducible von Neumann pair (A, B), then so does $(KMT)_2$. Summing up: in order to have the equality $\alpha^* = \beta^*$ it is sufficient that (under $(KMT)_1$ and $(KMT)_2$):

• The pair (A, B) is technologically irreducible.

• The pair (A, B) is economically irreducible.

• It holds $(A + B) > [0]$.

• More generally: for all $x \geq [0]$ such that supp(Ax) $\subset$supp(Bx) and for all $y \geq [0]$ such that supp$(y^\top B)$ $\subset$supp$(y^\top A)$ we have $y^\top Bx > 0$ or equivalently $y^\top Ax > 0$.

Other considerations on reducibility and irreducibility for a von Neumann technology (A, B) are made by Giorgi (2007), Groth (1986), Moczar (1991, 1995, 1997), Roemer (1980), Sanchez Choliz (1991), Weil (1970). Jaksch (1977) gives a necessary and sufficient condition to have $\alpha^* = \beta^*$, but it is a condition of a "quantitative nature", not of a "qualitative nature".

Remark 6. Robinson (1973) points out that a von Neumann technology may be reducible in both senses, technologically and economically, satisfy $(KMT)_1$ and $(KMT)_2$, even if the von Neumann condition $(A + B) > [0]$ holds true. This author gives the following simple example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This system is reducible, yet it satisfies the von Neumann condition.

Other conditions characterizing technological irreducibility are given by Robinson (1973):

• Let (A, B) be a von Neumann technology. Then the following properties are equivalent.

(i) (A, B) is technologically irreducible and $(KMT)_2$ holds.

(ii) There is at least a pair $(x, \alpha) \in \mathbb{R}^{n+1}$, such that $(B - \alpha A)x \geqq [0]$, with $x \geq [0]$, $\alpha > 0$, and for each such pair we have $Ax > [0]$.

(iii) $(KMT)_2$ holds and for each $\alpha > 0$ there exists a positive matrix $V(\alpha)$, of order m such that $V(\alpha)(B - \alpha A) < A$.

Similar characterizations hold with regard to economic irreducibility.

Remark 7. In the previous results we have obtained that under $(KMT)_1$ and $(KMT)_2$, both α^* and β^* exist and are positive (equal, if the model is irreducible, in a technological sense or in an economic sense, or if it satisfies the original von Neumann condition $(A+B) > [0]$). If we want to have $\alpha^* > 1$, $\beta^* > 1$, i. e. that the maximal growth *rate* is positive and that the minimal interest *rate* is positive, we have to assume that the matrix $(B-A)$ is *productive* (Nikaido (1968), Remark 3, page 147, speaks of "sufficiently productive technology"). The von Neumann technology (A, B) is productive if there is an intensity vector

$x^0 \geq [0]$ such that $(B - A)x^0 > [0]$. See also Giorgi and Meriggi (1987, 1988), Gale (1960), Fiedler and Ptàk (1966). Productive matrices are also called, in Matrix Theory, *S-matrices* or *matrices of the S-class.* It can be shown that the previous inequalities can be equivalently written as

$$\begin{cases} (B - A)x^0 > [0] \\ x^0 > [0] \,. \end{cases}$$

4. A generalization of the von Neumann model to a non polyhedral technology

David Gale (1957) and subsequently Samuel Karlin (1959) were concerned with a more general model of production than the original von Neumann model. See also Makarov and Rubinov (1977). We follow mainly Karlin (1959).

Consider an economy with n distinct goods where every good can be reproduced by the technology obeying the laws of non-increasing marginal rates of substitution and of constant returns to scale. The technology possibility of production is described by the transformation set T of the economy; namely, T is the set of pairs of the vectors, with n elements, (x, y) such that the production of the output

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

is technologically possible from the input

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if and only if $(x, y) \in T$.

It will be assumed that the transformation set T satisfies the following assumptions:

T_1) T is a closed convex cone in the nonnegative orthant of $\mathbb{R}^{2n}$.

T_2) $(x, y) \in T$ and $x' \geqq x$, $[0] \leqq y' \leqq y$ imply $(x', y') \in T$. This assumption states that the disposal activity is costless.

T_3) $([0], y) \in T$ implies $y = [0]$. That is, it is impossible to produce something from nothing.

T_4) For any $i = 1, .., n$, there exists $(x^i, y^i) \in T$ such that $y_i^i > 0$, where y_i^i is the i-th component of the vector y^i. In other words, every good can be produced.

Note that T_4), in view of T_1), is equivalent to

T'_4) There exists $(x^0, y^0) \in T$ such that $y^0 > [0]$.

For a possible nonzero input-output relation $(x, y) \in T$, the *rate of expansion* $\lambda(x, y)$ will be defined by

$$\lambda(x, y) = \max \{\lambda : y \geqq \lambda x\}.$$

By T_1) and T_3) we have

$$0 \leqq \lambda(x, y) < +\infty.$$

Indeed, we have the following result.

Theorem 4. There exists $(\bar{x}, \bar{y}) \in T$ such that

$$\bar{y} = \bar{\lambda}\bar{x}; \quad \bar{\lambda} = \lambda(\bar{x}, \bar{y}), \quad \bar{x} \geq [0]; \tag{18}$$

$$\bar{\lambda} \geqq \lambda(x, y) \text{ for any } (x, y) \in T, \ x \geq [0]. \tag{19}$$

Proof. Let be

$$\bar{\lambda} = \sup \{\lambda(x, y) : (x, y) \in T\}. \tag{20}$$

Then

$$0 < \bar{\lambda} < +\infty.$$

In fact, taking $(x^0, y^0) \in T$ with $y^0 > [0]$, we have

$$\bar{\lambda} \geqq \lambda(x^0, y^0) > 0.$$

If, on the other hand, $\bar{\lambda}$ is not finite, then there exists a sequence $\{(x^k, y^k)\}$ such that

$$\begin{array}{rcl} (x^k, y^k) & \in & T, \ y^k \geq [0], \\ y^k & \geqq & \lambda_k x^k, \\ \lim_{k \longrightarrow +\infty} \lambda_k & = & +\infty. \end{array}$$

Normalizing y^k such as $\sum_i y_{i,k} = 1$ and taking a limit point $\bar{y}$ of $\{y^k\}$, we have $([0], \bar{y}) \in T$ such that $\bar{y} \geq [0]$. This contradicts T_3). Since T is a closed cone, there exists $(\bar{x}', \bar{y}') \in T$ with $\bar{\lambda} = \lambda(\bar{x}', \bar{y}')$, where $\bar{\lambda}$ is defined by (20). By T_2), we can find $(\bar{x}, \bar{y}) \in T$ which satisfies (18) and (19). □

$(\bar{x}, \bar{y})$ is referred as a *balanced growth* and $\bar{\lambda}$ as the *maximal rate of the balanced growth.* Associated with the balanced growth $(\bar{x}, \bar{y})$, we have also an equilibrium price system, described by the following result.

Theorem 5. There exists a vector $\bar{p}$ such that $\bar{p} \geq [0]$ and

$$(\bar{p})^\top y \leqq \bar{\lambda}(\bar{p})^\top x, \ \forall (x, y) \in T. \tag{21}$$

Proof. Let be

$$C = \left\{ y - \bar{\lambda} x : (x, y) \in T \right\}.$$

Then C is a convex cone in $\mathbb{R}^n$ and has no interior point in common with the positive orthant. Therefore, by a result on convex cones ("separation property") there exists a semipositive vector $\bar{p}$ satisfying (21). □

Remark 8. The classical von Neumann model may be considered as a special case of the above model. The transformation set now becomes

$$T = \{(x, y) : x \geqq Az,\ y \leqq Bz \text{ for some } z \geqq [0]\}.$$

Assumption T_1) and T_2) are always satisfied. T_3) now becomes:

• For any j, there is at least one index i such that $a_{ij} > 0$ (every activity uses some good as input), i. e. $(KMT)_1$.

Assumption T_4) now becomes:

• For any index i there is at least one j such that $b_{ij} > 0$ (every good can be produced by some activity), i. e. $(KMT)_2$.

Theorem 5 combined with Theorem 4 implies the "kernel" of the classical von Neumann model: if the nonnegative matrices A, B satisfy the above conditions, then there exist an activity vector $\bar{z}$ and a price vector $\bar{p}$ such that

$$\begin{aligned} \bar{z} &\geq [0], \quad \bar{p} \geq [0], \\ B\bar{z} &\geqq \bar{\lambda} A \bar{z}, \\ (\bar{p})^\top B &\leqq \bar{\lambda} (\bar{p})^\top A. \end{aligned}$$

5. Additional remarks on the von Neumann model

I) **A Leontief-von Neumann model.**

When we have, in a von Neumann model, $m = n$ and $B = I$ (identity matrix), we can speak of a Leontief-von Neumann model. This subject has been treated by Gale (1960), Łos (1971) and, in a different context, by Morishima (1961, 1964). On the grounds of Definition 2 a triplet (p, v, α), with $p \in \mathbb{R}^n_+$, $v \in \mathbb{R}^n_+$ $\alpha > 0$, is an equilibrium solution of a Leontief-von Neumann technology if the following conditions are satisfied:

$$\alpha A v \leqq v \tag{22}$$

$$\alpha p^\top A \geqq p^\top \tag{23}$$

$$p^\top v > 0. \tag{24}$$

We continue to assume the validity of $(KMT)_1$ and $(KMT)_2$. Obviously, $(KMT)_2$ is automatically verified, being $B = I$. There exist strict relationships between the above equilibrium solutions and the positive eigenvalues of the matrix A.

Theorem 6. Let A be the technological matrix of a Leontief-von Neumann model.

a) If the triplet (p, v, α) is an equilibrium solution of the Leontief-von Neumann model, then $\frac{1}{\alpha}$ is an eigenvalue of the matrix A, with an associated semipositive (right-hand side) eigenvector $\hat{v}$.

b) If $\hat{v} \geq [0]$ is an eigenvector of the matrix A, associated with the eigenvalue $(1/\alpha) > 0$, then there exists a semipositive vector $p \in \mathbb{R}^n_+$ such that the triplet $(p, \hat{v}, \alpha)$ is an equilibrium solution of the Leontief-von Neumann model.

Proof.

a) Let the triplet (p, v, α) be an equilibrium solution of a Leontief-von Neumann model; hence (22), (23) and (24) are satisfied. Let us consider the following three sequences:

$$v,\ \alpha A v,\ \alpha^2 (A)^2 v, ..., \alpha^k (A)^k v, ... \tag{25}$$

$$p^\top,\ \alpha p^\top A,\ \alpha^2 p^\top (A)^2, ..., \alpha^k p^\top (A)^k, ... \tag{26}$$

$$p^\top v,\ \alpha p^\top v,\ \alpha^2 p^\top (A)^2 v, ..., \alpha^k p^\top (A)^k v, ... \tag{27}$$

where $(A)^k$ denotes the k-th power of A. From inequality (22) it results that the sequence (25) is nonincreasing, whereas (23) ensures that the sequence (26) is nondecreasing. The sequence (27) is at the same time nonincreasing and nondecreasing and it is therefore a constant. The sequence (25), having nonnegative elements, is convergent. Let be

$$\hat{v} = \lim_{k \longrightarrow +\infty} \alpha^k (A)^k v. \tag{28}$$

From (28) we obtain

$$\hat{v} = \lim_{k \longrightarrow +\infty} \alpha^k (A)^k v = \alpha \left(\lim_{k \longrightarrow +\infty} \alpha^{k-1} (A)^{k-1} v \right) = \alpha A \hat{v},$$

that is

$$A\hat{v} = \frac{1}{\alpha} \hat{v}. \tag{29}$$

Taking the limit for $k \longrightarrow +\infty$ in (27), taking (28), (24) into account and being the sequence (27) constant, we have

$$p^\top \hat{v} = p^\top v > 0, \quad \hat{v} \geq [0]\,. \tag{30}$$

Hence $(1/\alpha)$ is a positive eigenvalue of A, with $\hat{v}$ as a corresponding (right-hand side) eigenvector. We deduce therefore from (23), (29) and (30) that the triplet $(p, \hat{v}, \alpha)$ is an equilibrium solution of the Leontief-von Neumann technology.

b) Let $v \geq [0]$ be a (right-hand side) eigenvector of A corresponding to the eigenvalue $\frac{1}{\alpha} > 0$. Hence the following equality holds

$$\alpha A\hat{v} = \hat{v}.$$

Let us suppose that there exists no $p \in \mathbb{R}^n_+$ such that the triplet $(p, \hat{v}, \alpha)$ is an equilibrium solution of the Leontief-von Neumann model. By Lemma 4 it results the existence of a vector $v \in \mathbb{R}^n_+$ such that

$$\hat{v} \leqq v - \alpha A v,$$

that is

$$\hat{v} + \alpha A v \leqq v.$$

Multiplying both sides of this inequality by αA, and adding, we obtain

$$2\hat{v} + \alpha^2 (A)^2 v \leqq v.$$

By repeating k times this operation, it results

$$k\hat{v} + \alpha^k (A)^k v \leqq v.$$

Hence

$$k\hat{v} \leqq v$$

for every k. This inequality implies $\hat{v} = [0]$, which contradicts the assumption that $\hat{v}$ is an eigenvector of A. □

Remark 9. If A is irreducible in the usual sense (see, e. g., Debreu and Herstein (1953), Gantmacher (1959), Varga (1962)), then $v > [0]$, $p > [0]$ and $\alpha = a^* = \beta^*$. In other words, α is the Frobenius root of A and v is the right-hand side positive eigenvector corresponding to $(1/\alpha)$. The eigenvectors v and p are positive, unique, up to a multiplication by a positive number.

II) **The case of "cone inclusions".**

When both A and B are square of order n and suitable assumptions hold, some results of point *I*) can be fitted to a von Neumann model. This has been suggested by Łos (1971) and by Thompson and Weil (1971). For a more systematic treatment, see Abraham-Frois and Berrebi (1979) and Giorgi (2016). We begin with some interesting results of Mangasarian (1971) and Steenge and Konijn (1992) which hold for non necessarily square matrices. The following proposition is due to Mangasarian (1971).

Theorem 7. Let be given A and B, both of order (m, n). If

$$Bx \geqq [0] \Longrightarrow Ax \geqq [0], \tag{31}$$

then this implication is equivalent to the existence of a nonnegative matrix M, of order m, such that

$$A = MB.$$

If

$$Bx > [0] \Longrightarrow Ax > [0]\,, \tag{32}$$

then this implication is equivalent to the existence of a semipositive matrix M, of order m, with *all semipositive rows,* i. e. $M_i \geq [0]\,,\ i = 1, ..., m$, such that

$$A = MB.$$

In particular, if A and B are square of order n, and B is nonsingular, implication (31) is equivalent to

$$AB^{-1} \geqq [0]\,.$$

Note that in both implications (31) and (32) it is not assumed that A and B are nonnegative. If these matrices are nonnegative, then (31) or (32) mean that the ouput matrix cone is contained in the input matrix cone (the positive output matrix cone is contained in the positive input matrix cone). These conditions were also pointed out by Hicks (1965). If the converse of implication (31) holds, with $m = n$ and A nonsingular, we obtain the relation $BA^{-1} \geqq [0]\,.$

Theorem 8. (Steenge and Konijn). Let be given the von Neumann technology (A, B) and assume that $(KMT)_1$ and $(KMT)_2$ hold. Furthermore, assume that (32) is verified and that the nonnegative matrix M is irreducible (in the usual sense). Then the pair (A, B) is irreducible in the Gale sense and hence $\alpha^* = \beta^* > 0$.

We have seen that the von Neumann condition $(A + B) > [0]\,,$ also under $(KMT)_1$ and $(KMT)_2$, does not imply irreducibility in the sense of Gale (recall the counterexample of Robinson (1973)). The following proposition, due again to Steenge and Konjin (1992), established a link betwen the two conditions.

Theorem 9. Suppose that (A, B) satisfies $(KMT)_1$ and $(KMT)_2$ and that (32) is verified. Then, if $(A + B) > [0]\,,$ the system (A, B) is irreducible in the sense of Gale.

If both A and B are square, nonnegative, $(KMT)_1$ and $(KMT)_2$ hold (i. e. they form a von Neumann technology), and B is nonsingular, then, if (31) holds, we have, as previously pointed out,

$$M = AB^{-1} \geqq [0]\,.$$

This case, together with the cases $B^{-1}A \geqq [0]\,,\ A^{-1}B \geqq [0]\,,\ BA^{-1} \geqq [0]\,,$ has been studied, in a von Neumann growth model context, by Abraham-Frois and Berrebi (1979). Also Łos (1971) gives some hints, but without developing the subject. Steenge and Konijn (1992) assume that (32) holds and that $(KMT)_1$ and $(KMT)_2$ hold, with A, B of order (m, n). In their Proposition 5 the said authors establish a link between the equilibrium level α (for a von Neumann technology) and the Frobenius root $\lambda^*(M)$, where $\lambda^*(M) > 0$, thanks to (32). For the reader's convenience we give a sketch of the missing proof of Proposition 5 of Steenge and Konijn (1992).

Theorem 10. Let $(KMT)_1$ and $(KMT)_2$ be satisfied, together with relation (32). Let $Bx \geqq \alpha Ax$, with $\alpha > 0$, $x \geq [0]$. Then $\alpha \leqq 1/\lambda^*(M)$, being $\lambda^*(M) > 0$ the Perron-Frobenius eigenvalue of M.

Proof. For simplicity, let us assume that $Mz = \lambda^*(M)z$, where M is indecomposable, $z > [0]$ the Perron-Frobenius eigenvector and $\lambda^*(M) > 0$ the corresponding eigenvalue. It is important to note that there is only one such pair where the eigenvalue and eigenvector are both positive. As (32) holds we have

$$Bx \geqq \alpha Ax > [0], \quad \alpha > 0.$$

However, $A = MB$, so by substitution we have

$$Bx \geqq \alpha MBx.$$

Putting $y \equiv Bx$ we may write $y \geqq \alpha My$, with $y > [0]$ or $(1/\alpha)y \geqq My$. Now we try to establish the relation between α and $\lambda^*(M)$. Suppose

$$\alpha > (1/\lambda^*(M)), \text{ i. e. } (1/\alpha) < \lambda^*(M)$$

and suppose $y_j > M_j y$ for some j. Then definitely

$$y_j > (1/\lambda^*(M))M_j y$$

or

$$\lambda^*(M)y_j > M_j y.$$

Suppose, alternatively for some $i \neq j$,

$$y_i = \alpha M_i y;$$

then

$$y_i > (1/\lambda^*(M))M_i y$$

or

$$\lambda^*(M)y_i > M_i y.$$

So, combining we have

$$\lambda^*(M)y > My,$$

which is impossible because $\lambda^*(M)$ would not be the Perron-Frobenius eigenvalue. The same is true for two equalities, etc. So, $\alpha \leqq (1/\lambda^*(M))$. Really, as M is irreducible we have in fact $\alpha = \alpha^* = \beta^* = (1/\lambda^*(M)) > 0$. □

III) **A result of Hellwig and Moeschlin.**

K. Hellwig and O. Moeschlin (1974) point out the following considerations. We recall that, given a matrix split $M_\alpha = B - \alpha A$, with A and B nonnegative, of order (m, n) and satisfying $(KMT)_1$ and $(KMT)_2$, a triplet $(\bar{x}, \bar{y}, \bar{\alpha})$, $x \geq [0]$, $\bar{y} \geq [0]$, $\bar{\alpha} > 0$, is termed an *equilibrium solution* of the related von Neumann technology if

$$\begin{aligned} M_{\bar{\alpha}}\bar{x} &\geqq [0] \\ (\bar{y})^\top M_{\bar{\alpha}} &\leqq [0] \\ (\bar{y})^\top B\bar{x} &> 0. \end{aligned}$$

We have proved in the previous sections the existence of equilibrium solutions of the said relations and other formal properties. In particular, we have proved the existence of an equilibrium solution (x^*, y^*, α^*), where α^* solves the following optimization problem

$$Q(\alpha): \qquad \max_{\alpha, x} \{M_\alpha x \geqq [0], \ x \geq [0]\}.$$

This result states, from an economic point of view, that the highest possible growth factor is an equilibrium growth factor. Hellwig and Moeschlin (1974) consider the following optimization problem

$$\begin{aligned} & \max \alpha \\ \text{subject to:} \quad & M_\alpha x \geqq [0] \\ B_j x &> 0, \quad j \in \bar{J} \\ x &\geqq [0] \end{aligned} \tag{33}$$

where B_j denotes the j-th row of B and $\bar{J}$ is a subset of $J \equiv \{1, ..., m\}$.

Surely, a solution to (33) need not exist, but if it exists, we have the following interesting result (the condition $B_j x > 0$, $j \in \bar{J}$, ensures that at the maximal growth factor of (33) certain goods, i. e. the ones indicated by $\bar{J}$ can be produced).

Theorem 11. Assume that the pair (A, B) satisfies the assumptions $(KMT)_1$ and $(KMT)_2$. If there exists a solution $(\hat{x}, \hat{\alpha})$ to (33), $0 < \hat{\alpha} < +\infty$, then there exists an equilibrium solution $(\bar{x}, \bar{y}, \bar{\alpha})$, where $\bar{\alpha} = \hat{\alpha}$.

The proof of Theorem 11 is essentially performed by means of Lemma 3 and Lemma 4 and parallels the proof of Theorem 1. We refer the reader to the paper of Hellwig and Moeschlin (1974).

IV) **Proof of the existence of the equilibrium solution by means of the Kuhn-Tucker theory.**

T. Fujimoto (1975) presents a proof of the existence of equilibrium solutions in the von Neumann model by use of the Kuhn-Tucker theory. It seems that this method of proof was suggested by Morishima (1969). However, see Kuhn (1968), where this method is explicitly used. Fujimoto (1975) takes into consideration the classical von Neumann relations

$$\begin{aligned}
(1)\ \ Bx &\geqq \alpha Ax \\
(2)\ \ p^\top B &\leqq \beta p^\top A \\
(3)\ \ p^\top Bx &= \alpha p^\top Ax \text{ ("rule of free goods")} \\
(4)\ \ p^\top Bx &= \beta p^\top Ax \text{ ("rule of profitability").}
\end{aligned}$$

Note that the condition $p^\top Bx > 0$ is not imposed, however it comes out from the proof. Fujimoto assumes that the conditions $(KMT)_1$ and $(KMT)_2$ are verified. Then this author considers the following problem

$$\max \alpha, \text{ subject to equation (1) and } e^\top x = 1.$$

First form the Lagrangian function of this problem:

$$\mathcal{L} = \alpha - \mu^\top(\alpha Ax - Bx) - \lambda(e^\top x - 1).$$

(Obviously $\mu^\top$ is a row vector and λ is a scalar). By the assumption, there is at least a pair (α^*, x^*), $\alpha^* > 0$, $x^* \geq [0]$, which is a solution, unique or not unique. We can apply the Kuhn-Tucker theorem. At the solution point, we can find μ^* and λ^* such that

$$1 - (\mu^*)^\top Ax^* = 0, \tag{34}$$

$$-(\mu^*)^\top \alpha^* A + (\mu^*)^\top B - \lambda^* e \leqq [0], \tag{35}$$

$$-(\alpha^* Ax^* - Bx^*) \geqq [0], \tag{36}$$

$$((-\mu^*)^\top \alpha^* A + (\mu^*)^\top B - \lambda^* e)x^* = 0, \tag{37}$$

$$-(\mu^*)^\top(\alpha^* Ax^* - Bx^*) = 0. \tag{38}$$

From (37) and (38) we have $\lambda^* = 0$, since $e^\top x^* = 1$. This implies that the constraint $e^\top x = 1$ is inessential. Now we can write (35)-(38) as follows:

$$(\mu^*)^\top B \leqq \alpha^*(\mu^*)^\top A,$$

$$Bx^* \geqq \alpha^* Ax^*,$$

$$(\mu^*)^\top Bx^* = \alpha^*(\mu^*)^\top Ax^*.$$

Thus, by putting $\alpha = \beta = \alpha^*$, $x = x^*$, $p = \mu^*(\geq [0]$ by (34)), in the von Neumann system considered above, we obtain an equilibrium set of variables. Equations

$$\begin{aligned} p^\top Bx &= \alpha p^\top Ax \\ 1 - p^\top Ax &= 0 \end{aligned}$$

tell that the value of the output is positive, i. e. $p^\top Bx > 0$. □

Alternatively we could formulate the minimization problem

$$\min \beta \ \text{ subject to } p^\top B \leqq p^\top A,\ p^\top e = 1.$$

In a similar way we can find an equilibrium where the minimum interest factor is observed.

6. Some conclusions

We have reviewed the basic mathematical properties of the classical von Neumann growth model, properties often scattered in several papers and books, sometimes with scarse evidence on the various links between the said properties. Again we put into consideration that the original von Neumann model can be related to a pair of optimization problems, i. e. the problems

$$Q(\alpha): \qquad \begin{cases} \max\limits_{\alpha,x} \{(B - \alpha A)x \geqq [0]\} \\ \qquad x \geq [0]\,, \quad \alpha > 0 \end{cases}$$

$$P(\beta): \qquad \begin{cases} \min\limits_{\beta,y} \{y^\top (B - \beta A) \leqq [0]\} \\ \qquad y \geq [0]\,, \quad \beta > 0. \end{cases}$$

This fact has been put into evidence by von Neumann himself, together with its implication on the symmetry and duality of the model. It must be pointed out that, for example, Kemeny, Morgenstern and Thompson (1956) apparently ignore the possibility to associate the model with a pair of optimization problems, whereas Gale (1960), on the contrary, is essentially concerned with the said problems. If the assumptions $(KMT)_1$ and $(KMT)_2$ hold and the pair (A, B) is irreducible in the sense of Gale or of Robinson or if the assumptions $(KMT)_1$ and $(KMT)_2$ hold and $(A + B) > [0]$, then the solutions of the original von Neumann model coincide with the solutions of $Q(\alpha)$ and $P(\beta)$. Needless to say that the von Neumann model has been one of the first formal economic models to study the optimal development of a multi-sector economy. Morishima (1969) has spoken of a "von Neumann revolution" and Goodwin (1985) has confessed that "the greatest single intellectual mistake in my career occurred when Schumpeter came to me in 1938 or '39 and asked me to report on a very important new publication: the von Neumann paper given at the Menger seminar...

I rashly judged it to be totally unrealistic... and reported back to Schumpeter that it was no more than a piece of mathematical ingenuity, failing to see that it contained two aspects close to Schumpeter's heart - a rigorous solution to Walras' central problem and a demonstration that the rate of profit arose from growth not a quantity of capital... I found no reference in Schumpeter's *History* to what now appears to me to be one of the great seminal works in this century, the omission being possibly the result of my own blindness".

Obviously the literature on the von Neumann model and its variants and generalizations is huge. Besides the works quoted in the present paper, the reader may take profit by the ones of Afriat (1974, 1987a, b), Aubin (1993), Bauer (1975), Bidard and Hosoda (1987), Bose and Bose (1972), Burmeister and Dobell (1970), Dore, Chakravarty and Goodwin (1989), Haga and Otsuki (1965), Łos (1974, 1976), Łos, Łos and Wieczorek (1976), Moeschlin (1974), Morishima (1964, 1969), Makarov and Rubinov (1977), Nikaido (1970). Schefold (1980), Ye (1995), Woods (1978), Zalai (2004).

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