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Coherence without Rationality at the ZLB

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Coherence without Rationality at the ZLB[*]

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Abstract

Standard rational expectations models with an occasionally binding zero lower bound constraint either admit no solutions (incoherence) or multiple solutions (incompleteness). This paper shows that deviations from full-information rational expectations mitigate concerns about incoherence and incompleteness. Models with no rational expectations equilibria admit self-confirming equilibria involving the use of simple mis-specified forecasting models. Completeness and coherence is restored if expectations are adaptive or if agents are less forward-looking due to some information or behavioral friction. In the case of incompleteness, the E-stability criterion selects an equilibrium.

Keywords: incompleteness, incoherence, expectations, zero lower bound
JEL classification: C62: E4: E52

The irrationality of a thing is no argument against its existence, rather a condition of it.
Friedrich Nietzsche, "Human, All Too Human: A Book for Free Spirits", 1878.

1   Introduction

In the last 15 years after the Great Financial Crisis, central banks in western economies had to face the problem of a zero (or effective) lower bound (ZLB) on the nominal interest rate. This spurred a very large and important literature on the topic. At least from the seminal contribution by Benhabib et al. (2001), it is well-known that rational expectations (RE) models with a ZLB on the nominal interest rate generally admit multiple equilibria and also multiple steady states. However, the stochastic element in the ZLB literature is often very stylized with one single (often discount factor) shock that occurs only once and has either a stochastic or a known duration.

Most recently, however, Ascari and Mavroeidis (2022, henceforth AM) highlight an even more serious concern regarding this type of models when stochastic shocks hit the economy, a standard assumption in macroeconomic models. They show that in models featuring a ZLB constraint, a stochastic environment and RE, equilibrium existence is not generic, i.e., the model is incoherent, and when these model do admit an equilibrium, they generally admit more equilibria than previously acknowledged, i.e., the model is incomplete.[1] Specifically, AM derive conditions for existence of a rational expectations equilibrium (REE), and for existence and uniqueness of a minimum state variable (MSV) equilibrium for dynamic forward-looking models with occasionally binding constraints. These conditions are difficult to interpret. Therefore, AM highlight a different and more fundamental problem in models with occasionally binding constraints and standard stochastic shocks than the ones already noted in the literature in this class of models, such as the indeterminacy of REE equilibria in linear models and/or multiplicity of steady states. Section 3 reviews the AM results in more detail.

Given that a model without an equilibrium could not be of any use, this paper points to a possible route to tackle the incoherence problem: abandoning the full-information RE assumption. We show that the problem of incoherence and incompleteness hinges on the assumption that agents have RE. Non-existence of REE is by itself a compelling and novel reason to investigate the possibility of non-rational equilibria. Indeed, one of the main results from this paper is that a standard New Keynesian model with the ZLB constraint can fail to yield an REE and still admit other types of self-confirming equilibria. To illustrate this point, we consider two distinct equilibrium concepts which have been associated with

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[1]Following AM we will use the terms incoherence and incompleteness to mean the non-existence of equilibria and the multiplicity of equilibria, respectively. Hence, a model is coherent if it admits at least one equilibrium, and complete if the equilibrium is unique.

different types of deviations from full-information RE.

First, we investigate one of the most studied deviations from RE, that is, adaptive learning as typified by Evans and Honkapohja (2001). Adaptive learning agents have imperfect knowledge about the economy's structure, but learn to forecast macroeconomic variables by recursively estimating the parameters of a subjective forecasting model using simple statistical tools like least squares. A classic question examined in adaptive learning applications is whether agents eventually learn to forecast rationally, and hence whether the learning economy converges to an REE. However, given that we are interested in cases where an REE does not exist, we assume that agents learn by recursively estimating forecasting models that are mis-specified and under-parameterized relative to the forecasting models that agents would have in an REE. Under this assumption, we derive analytically conditions for the economy to settle on a self-confirming equilibrium in which agents make optimal forecasts within their class of forecasting rule. This form of self-confirming equilibrium, which is distinct from REE, is often labelled *restricted perceptions equilibrium (RPE)* in the learning literature (e.g. see Evans and Honkapohja (2001)). Importantly, we prove that an RPE can exist when the RE model is incoherent and hence no REE exists.

Second, we consider bounded rationality as a possible deviation from RE. Boundedly rational agents are less forward-looking than rational agents, for instance because they are myopic à la Gabaix (2020), have imperfect common knowledge as in Angeletos and Lian (2018), or have finite planning horizons similar to Woodford and Xie (2020). In this setting too, a unique *bounded rationality equilibrium (BRE)* may exist, even if an REE does not. Hence, both adaptive learning and bounded rationality might alleviate, under certain conditions, the coherence problem of the standard NK model with a ZLB constraint. Finally, we also investigate the implications of combining the two deviations from rationality.

The derivation of an adaptive learning RPE and BRE and in an incoherent REE framework is the central contribution of the paper. In this respect, some remarks are noteworthy.

First, adaptive learning can ensure completeness and coherence all by itself. Specifically, we prove that a unique *temporary* equilibrium always exists in our model with a ZLB constraint and adaptive learning agents, provided that agents do *not* observe current endogenous variables before market clearing takes place – a very common assumption in the learning literature. If learning agents condition their forecasts on current information about endogenous variables, then a temporary equilibrium only exists under more stringent assumptions.

Second, the learning literature has typically focused on the question of whether an REE can be learnable, because the underlying model admits an REE solution. Here, instead, we want to investigate if adaptive learning could generate equilibria even when an REE does not exist. In the case agents do *not* observe current endogenous variables, expectations are predetermined, and a temporary equilibrium always exists, but it is not necessarily self-confirming. However, we show that an RPE emerges as a self-confirming equilibrium, even if

the underlying model does not admit an REE. To the best of our knowledge, the existence of a self-confirming adaptive learning equilibria when there is no REE is a novel and intriguing case in the literature.

Third, and related to the previous point, whenever the NK model does not admit an REE, it is impossible for agents to form self-confirming beliefs about the dynamics of inflation and output (i.e., as implied by a standard MSV in our simple model). The economy can easily diverge into a deflationary spiral if agents attempt to learn these dynamics using simple statistical techniques. Hence, while it is a curse to be smart, it is a blessing to be simple-minded, because the non-rationality of agents' beliefs can save the economy from spiralling out of control and lead it to a coherent and complete (CC) self-confirming RPE.

Fourth, the source of the problem of rational incoherence can be intuitively explained in terms of income and substitution effects, following Bilbiie (forth.). A similar intuition is behind the so-called "forward guidance puzzle" and its proposed solutions that hinge on weakening agents' forward-lookingness (e.g., Del Negro et al., 2012; McKay et al., 2016b; Angeletos and Lian, 2018; Gabaix, 2020; Woodford and Xie, 2020; Eusepi et al., 2021). Hence, we show that weakening the 'rationality' of agents seems to kill a few birds with one stone, because it simultaneously solves different problems highlighted by the literature (forward-guidance puzzle, belief-driven liquidity traps, existence of an equilibrium, i.e., coherence) that share the same mechanism as a common source.

Fifth, a basic takeaway from the existence analysis is that the baseline NK model with RE is incoherent *when negative shocks are sufficiently large in magnitude or sufficiently persistent*, but can still admit RPE or BRE. A fundamentals-driven RE liquidity trap, thus, must be relatively short-lived compared to the duration of actual liquidity trap events experienced by Japan, the Euro Area and the U.S., because persistent shocks would make the RE model incoherent. This is not true for the RPE or BRE, where a liquidity trap can be highly persistent. In this sense, one could argue that an RPE or a BRE could explain why the economy did not blow up after a large shock as the Great Financial Crisis.

Finally, a second contribution of the paper concerns the stability properties of these equilibria under learning, that is, the issue of whether RPE and REE can emerge from a process of learning. Following the adaptive learning literature, we employ the expectational stability or "E-stability" criterion to select an equilibrium that may arise through an economy-wide adaptive learning process in which agents recursively update the parameters of their subjective forecasting models using simple statistical techniques such as least squares. We find there is a unique E-stable RPE when an RPE exists. Similarly, only one MSV REE can be E-stable.

After a brief literature review, the paper proceeds as follows. Section 2 introduces a simple model of the ZLB that nests our different assumptions about expectations formation as special cases. Section 3 illustrates the problem of rational incoherence and the possibility

of irrational coherence. Section 4 shows how adaptive learning resolves incompleteness issues, and also discusses the plausibility of the RPE concept. Section 5 suggests an additional route to irrational coherence: lagged information about economic shocks. Section 6 concludes. The proofs of all the Propositions can be found in the Appendix.

1.1 Literature review

This paper contributes to an already large literature about deviations from RE and the ZLB. Earlier work on adaptive learning at the ZLB studied monetary and fiscal policies that can prevent an economy with learning agents from getting stuck in the liquidity trap (Evans et al., 2008; Benhabib et al., 2014; Evans et al., forth.),[2] unconventional policies such as forward guidance (Cole, 2021; Eusepi et al., 2021), make-up strategies such as price level targeting (Honkapohja and Mitra, 2020) or average inflation targeting (Honkapohja and McClung, 2021). Christiano et al. (2018) show that the E-stability criterion selects one of multiple equilibria of a model with a transitory demand shock that can drive the economy into a liquidity trap. This finding is closely related to our result about E-stability of REE in the case of incompleteness. However, their model assumes that the economy returns to a steady state after the shock dissipates, whereas our framework allows for multiple, recurring liquidity trap episodes, consistent with the recurrence of ZLB events in the U.S. and elsewhere. Thus, we extend insights from Christiano et al. (2018) to models with recurring demand shocks. More generally, the above mentioned papers do not consider existence and stability of equilibria of models with recurring, fundamentals-driven liquidity traps.

A significant strand of the adaptive learning literature focuses on self-confirming "misspecification equilibria" that can emerge if agents recursively learn to forecast using a misspecified forecasting rule. In a misspecification equilibrium, agents do not understand the true equilibrium law of motion for economic variables, but observable macroeconomic outcomes nonetheless confirm their subjective beliefs about specific statistical properties of the economy. RPE is a special case of misspecification equilibrium involving a "simple" underparameterized forecasting model that omits some variables which affect the macroeconomic dynamics. In an RPE, agents forecast optimally within their class of forecasting rules in the sense that forecast errors are orthogonal to their forecasting model. The properties of RPE and misspecification equilibria, as well as their emergence through adaptive learning, has been explored in Branch (2006), Evans and Honkapohja (2001), Marcet and Sargent (1989), Evans et al. (1993), Branch and Evans (2006a), Branch and Evans (2006b), Bullard et al. (2008), Evans and McGough (2020) and Evans et al. (2021), Hommes and Sorger (1997), Hommes and Zhu (2014), among many others. Empirical support for RPE and related misspecification equilibria comes from experiments involving monetary sticky price economies

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[2]See also Evans and McGough (2018b) for a related discussion on interest rate pegs and adaptive learning.

(Adam, 2007) and analysis of survey and macroeconomic data involving estimation of New Keynesian frameworks (Hommes et al., forth.).[3]

A number of earlier works, including Angeletos and Lian (2018), Gabaix (2020) and Woodford and Xie (2020), study bounded rationality equilibrium and issues related to the ZLB. Among other things, these papers show that deviations from RE that make agents less forward-looking than rational agents can resolve the so-called NK paradoxes of the ZLB, such as the prediction that forward guidance announcements can have arbitrarily large effects on the economy ("forward guidance puzzle"). Importantly, contributions to this literature typically treat the ZLB regime as arising from a transitory shock, usually with a known duration, after which time the economy returns to steady state forever. Models employing shocks with known duration are not susceptible to the issues of equilibrium existence and multiplicity that we study here. Our contribution, therefore, is to embed bounded rationality into models with recurring stochastic shocks, and to show that these deviations from RE resolve the problem of incoherence and incompleteness identified by AM.

Finally, Mertens and Ravn (2014), Nakata and Schmidt (2019a, 2020), and Bilbiie (forth.), among others, study conditions for the existence of both fundamentals-driven and confidence-driven liquidity trap equilibria, which are caused by fundamental shocks to the economy and non-fundamental (sunspot) shocks, respectively.[4] One takeaway from these papers is that fundamentals-driven liquidity trap equilibrium is unlikely to exist if shocks are too persistent, but sunspot equilibria can feature very persistent liquidity traps. However, to our knowledge, confidence-driven liquidity trap equilibria have only been derived in coherent models (i.e. models that admit at least one MSV solution). An incoherent model can fail to admit confidence-driven liquidity trap equilibria, and tight restrictions on the support of *fundamental* shocks are necessary for existence of both MSV and confidence-driven liquidity trap equilibria.

2 Model and expectations formation mechanisms

We employ a model that nests the simple New Keynesian model as well as reflects the reduced-form of the alternative bounded rationality models explored by Gabaix (2020), Angeletos and Lian (2018), Woodford and Xie (2020):

$$x_t = M\hat{E}_t x_{t+1} - \sigma(i_t - N\hat{E}_t \pi_{t+1}) + \epsilon_t, \tag{1}$$

$$\pi_t = \lambda x_t + M_f \beta \hat{E}_t \pi_{t+1}, \tag{2}$$

$$i_t = \max\{\psi \pi_t, -\mu\}, \tag{3}$$

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[3]See also Slobodyan and Wouters (2012), Ormeno and Molnár (2015), Beshears et al. (2013), Assenza et al. (2021) for additional empirical support for small misspecified forecasting rules.

[4]Additionally, Bianchi et al. (2021) study implications of fundamentals-driven liquidity traps in a nonlinear New Keynesian model.

where x_t is the output gap, i_t the nominal interest rate and π_t is the inflation rate. If $M = N = M_f = 1$, the model nests the simple three-equations New Keynesian model of Woodford (2003) where (1) is the Euler equation, (2) is the NK Phillips Curve and (3) the monetary policy rule, described by the simplest Taylor rule but with a ZLB constraint. The model is log-linearized around the zero inflation steady state and $0 < \beta < 1$, $0 < \sigma, \lambda, \mu$, and $\psi > 1$ (i.e. the "Taylor principle" holds). Bounded rationality implies, instead, $0 < M, N, M_f \leq 1$. Note that $\hat{E}$ denotes (possibly non-rational) expectations and $\hat{E} = E$ denotes model-consistent expectations.

We follow earlier work, including Eggertsson and Woodford (2003), Nakata and Schmidt (2019a), Christiano et al. (2018), and AM, and assume that the demand shock, ϵ_t, follows a 2-state Markov process with transition matrix:

$$K = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

with $0 < p = Pr(\epsilon_t = \epsilon_1 | \epsilon_{t-1} = \epsilon_1) \leq 1$, $0 < q = Pr(\epsilon_t = \epsilon_2 | \epsilon_{t-1} = \epsilon_2) \leq 1$. If we assume $q = 1$ and $\epsilon_2 = 0$, similar to Eggertsson and Woodford (2003) or Christiano et al. (2018), then we have a model in which a transitory shock, $\epsilon_t = \epsilon_1 \neq 0$, displaces the economy from steady state, but the economy eventually returns to the absorbing steady state of the model when $\epsilon_t = \epsilon_2 = 0$. In the standard RE version of the model there are two non-stochastic steady states: one with zero inflation, and one with zero nominal interest rates. However, equilibrium inflation and output in the temporary state ($\epsilon_t = \epsilon_1$) depends on whether agents have full-information RE or whether they are boundedly rational in some way.

We consider three models of expectations formation. First, agents have full-information RE in the special case of the model with no discounting in the Euler equation and Phillips curve (1)-(3) and model consistent expectations.

Definition 1 *Agents have **full-information rational expectations (RE)** if and only if $\hat{E} = E$ and $M = M_f = N = 1$ in the NK model given by Equations (1)-(3).*

An REE, defined in Section 3, is a solution of the model (1)-(3) obtained under these assumptions. In keeping with the literature, we treat full-information RE as the benchmark model of expectations formation, against which we compare ZLB dynamics under alternative expectations formation mechanisms. Particular attention is paid to the possibility that agents do not have full knowledge about the structure of the economy, and consequently expectations can be model-inconsistent (i.e., $\hat{E} \neq E$). The adaptive learning literature in particular studies agents with imperfect knowledge who learn to forecast the law of motion for aggregate variables using standard statistical tools like least squares. In this setting, imperfect knowledge can imply model-inconsistent expectations, but the focus of a large swath of this literature is whether agents can form self-confirming beliefs, either by learning an REE, or some non-rational, self-confirming equilibrium if their subjective forecasting models

are mis-specified with respect to the rational forecasting models. Holding fixed the structure of the model, imperfect knowledge by itself can lead us to new insights about policy and macroeconomic dynamics.

Definition 2 *Agents have **imperfect knowledge** if $\hat{E} \neq E$; $M = M_f = N = 1$ in the NK model given by Equations (1)-(3)*

Of course, we can deviate from RE without breaking the assumption that agents have full knowledge about the structure of their economic environment. For instance, Gabaix (2020) derives a model in which households and firms are relatively myopic due to cognitive limitations. In this setting, myopia implies a change in the model structure in the form of discounting in the aggregate demand curve (1) (i.e., $M < 1$) and additional discounting in the Phillips curve (2) (i.e. $M_f < 1$). However, nothing in Gabaix's (2020) model prevents agents from having full knowledge about the world they inhabit, and therefore nothing prevents these boundedly rational agents from having model-consistent expectations. Hence, Gabaix's (2020) behavioral model shows how we can deviate from full-information RE without sacrificing the assumption that agents have perfect knowledge. Bounded rationality models by Angeletos and Lian (2018) and Woodford and Xie (2020) may also lead to reduced-form structural models with additional discounting in the structural equations. If M, M_f or N is less than one, we say that agents are boundedly rational.

Definition 3 *Agents are said to be **boundedly rational** if and only if $\hat{E} = E$ and $min\{M, M_f, N\} < 1$.*

3 Coherence: existence of an equilibrium

This Section investigates the problem of coherence, that is, of the existence of an equilibrium, under the three models of expectations formation just described.

To put the whole paper into context, it is worth clarifying the main contributions of AM. While the stochastic element in the literature on ZLB is often very stylized, featuring one single (often discount factor) shock that occurs only once and has either a stochastic or a known duration, AM consider the general problem of the conditions for existence and uniqueness of equilibria in dynamic forward-looking models with RE when some variables are subject to occasionally binding constraints, like in the ZLB case, and when recurrent stochastic shocks hit the economy, a standard assumption in macroeconomic models. AM propose to use a method based on Gourieroux et al. (1980) that studied this problem in the context of simultaneous equations models with endogenous regime switching, and derived conditions for existence and uniqueness of solutions, which Gourieroux et al. (1980) label as 'coherency and completeness' (CC) conditions. The problem of existence of equilibria,

i.e., coherence, in more standard stochastic environments commonly used in macroeconomic models is obviously fundamental and a first-order concern for this literature.[5]

There are two main takeaways from AM. First, the question of coherence is a nontrivial problem in models with a ZLB constraint and AM were able to provide some general results only for a limited class of models. A typical New Keynesian (NK) model with a ZLB constraint is not generically coherent both when the Taylor rule is active and when monetary policy is optimal under discretion. The restrictions on the support that are needed to restore an equilibrium are difficult to interpret because they are asymmetric and because they depend both on the structural parameters and on the past values of the state variables. AM show that the assumption of orthogonality of structural shocks is incompatible with coherence, because if a model admits multiple shocks, their support restrictions cannot be independent from each other. Second, imposing the (somewhat awkward) support restrictions needed to guarantee existence of a solution causes another serious problem: multiplicity of MSV solutions, i.e., incompleteness.[6] AM show the existence of many MSV solutions, possibly up to 2^k MSV equilibria, where k is the number of (discrete) states that the exogenous variables can take, for example, using a k-state approximation of an AR(1) process. While the literature on the ZLB recognized the possibility of multiple steady states and/or multiple equilibria, and of sunspots solutions due either to indeterminacy or to belief-driven fluctuations between the two steady states, this is a novel source of multiplicity, that concerns 'fundamental' solutions, i.e., MSV ones. This is particularly relevant because numerical solution algorithms usually search for a solution of this type. The multiplicity of MSV solutions arises from the interaction between RE and the non-linear nature of the problem, as we will show below. Our paper investigates whether relaxing the full-information RE assumption could alleviate the problems highlighted by AM by breaking this interaction.

3.1 Rationality without Coherence

We start by assuming full-information RE to illustrate the problem of incoherence. For simplicity, we focus on MSV REE, but some of the insights from our paper can be extended

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[5]Despite there is a large and expanding literature on solution algorithms for such models, (see e.g., Fernández-Villaverde et al., 2015; Guerrieri and Iacoviello, 2015; Gust et al., 2017; Aruoba et al., 2018, 2021; Eggertsson et al., 2021), there are no general conditions for existence of equilibria for this class of models, as say, the Blanchard-Kahn conditions for standard linear dynamic RE models. Moreover, NK models with a ZLB are often presented as (log)linear approximations around an equilibrium of some originally nonlinear model, whose existence needs to be checked being an obvious precondition of the analysis. A number of theoretical papers provide sufficient conditions for existence of MSV equilibria in NK models (see Eggertsson, 2011; Boneva et al., 2016; Armenter, 2018; Christiano et al., 2018; Nakata, 2018; Nakata and Schmidt, 2019b), while AM provide both necessary and sufficient conditions that can be applied more generally.

[6]Therein, an MSV equilibrium is defined as usually intended, that is, as a function of the state variables of the model. However, an incoherent model could in principle admit other types of equilibria, but, to the best of our knowledge, no work in the literature, including AM, has found them. We use the terminology MSV and REE interchangeably in the case of incoherence.

to study non-fundamental "sunspot" equilibria which feature extraneous volatility. Since our model, (1)-(3), is a purely forward looking model with a 2-state discrete-valued exogenous shock, the MSV REE law of motion for $Y_t = (x_t, \pi_t)'$ will assume the form $Y_t = \mathbf{Y}_j$ where $Y_t = \mathbf{Y}_1$ if $\epsilon_t = \epsilon_1$ and $Y_t = \mathbf{Y}_2$ otherwise.

Definition 4 *Rational expectations equilibrium (REE).* $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$ *is a rational expectations equilibrium if and only if* $\mathbf{Y}_j$ *solves* (1)-(3) *given* $\hat{E}_t(Y_{t+1}|\epsilon_t = \epsilon_j) = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t = \epsilon_j)\mathbf{Y}_1 + Pr(\epsilon_{t+1} = \epsilon_2|\epsilon_t = \epsilon_j)\mathbf{Y}_2$ *and* $\epsilon_t = \epsilon_j$ *for* $j = 1, 2$.

There are up to four MSV REE of (1)-(3). First, there is a possible solution in which interest rates are always positive ("PP" solution). Then, there is a potential solution with binding ZLB if and only if $\epsilon_t = \epsilon_1$, which we refer to as the "ZP" solution. Analogously, there could be a "PZ" solution with binding ZLB if and only if $\epsilon_t = \epsilon_2$. Finally, it is possible that the ZLB is always binding ("ZZ" solution). We add a superscript i to $\mathbf{Y}$ to distinguish between the REE (i.e. $\mathbf{Y}^i$ where $i = PP, ZP, PZ, ZZ$). Following AM, if at least one of the four possible REE exist then the model is coherent.

Proposition 1 *Consider* (1)-(3) *and suppose* $M = M_f = N = 1$, $\epsilon_2 \geq 0$. *An REE exists if and only if* $\epsilon_1 \geq \bar{\epsilon}_{REE}$, *where* $\bar{\epsilon}_{REE}$ *is a constant that depends on the model's parameters, defined in Equation* (A3) *in Appendix* A.1.

Proposition 1 generalizes Proposition 5 of AM to the case with $q < 1$. It establishes that under the conventional assumption that the Taylor rule (3) satisfies the Taylor Principle and recurrent demand shocks, we need to restrict the magnitude of the shocks, ϵ_t, to get an REE. For a solution to exist, ϵ_1 cannot be too negative (i.e. the shock cannot be too "big", in absolute value). The lower bound on ϵ_1, denoted as $\bar{\epsilon}_{REE}$, is increasing in p for standard parameters, which means that a model with more persistent shocks requires tighter restrictions on the magnitude of the shocks for an equilibrium to exist. This explains why fundamentals-driven liquidity trap cannot be persistent in an REE. A "big" shock is needed to take the economy into a liquidity trap, but then, for an REE to exist, it cannot be persistent. Thus, the model is not generically coherent; solutions only exist for special calibrations of the shock process and solutions do not exist if the shocks are too persistent (i.e. p is very high) or if the shock is big (ϵ_1 is very low).

Intuition from a special case. While Proposition 1 deals with the case with $q < 1$, the assumption that the high demand state is absorbing ($q = 1$) and equal to zero ($\epsilon_2 = 0$) is helpful for intuition.[7] Under this assumption, the economy under full-information RE either

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[7]The assumption $q = 1$ is standard in the literature (e.g., Eggertsson and Woodford, 2003; Christiano et al., 2018; Bilbiie, forth.). To explain the intuition, we borrow heavily from AM and Bilbiie (forth.).

returns to the steady state with zero inflation (i.e. $\pi_t = x_y = i_t = 0$) or the steady state with zero interest rates (i.e. $i_t = -\mu$, $\pi_t = -\mu < 0$ $x_t = -\mu(1-\beta)/\lambda < 0$). The "temporary state" value of output when $\epsilon_t = \epsilon_1 < 0$ (assuming for brevity that we go back to the zero inflation steady state) is given by:

$$x_t = \nu(p)E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-\beta p}x_t, -\mu\} + \epsilon_1, \tag{4}$$

$$\nu(p) = \left(1 + \frac{\lambda\sigma}{1-\beta p}\right) > 1, \tag{5}$$

which we obtain by substituting the Phillips curve and Taylor rule into (1). From (4), it is apparent that for any p, sufficiently low values of ϵ_1 preclude unconstrained interest rates. Thus, for a sufficiently large demand shock, output will be given by:

$$x_t = \frac{1}{1-p\nu(p)}(\sigma\mu + \epsilon_1) \tag{6}$$

if a solution of the model exists at all. However, if the negative demand shock is sufficiently persistent, so that $p\nu(p) > 1$, then x_t and therefore temporary inflation, $\pi_t = \frac{\lambda}{1-\beta p}x_t$ are decreasing in ϵ_1. This implies that sufficiently large ϵ_1 will increase x_t and π_t, precluding existence of a solution in which the ZLB binds. Therefore, for a solution to exist we need to do one of two things. First, we can restrict p to be small enough to ensure $p\nu(p) < 1$, which in turn implies a solution for any ϵ_1. Or, alternatively, we need to restrict ϵ_1 to be small (i.e. close to zero) to rule out a situation where large demand shocks preclude both unconstrained and constrained equilibrium interest rates – i.e., incoherence. Both options require restrictions on the support of the demand shock.

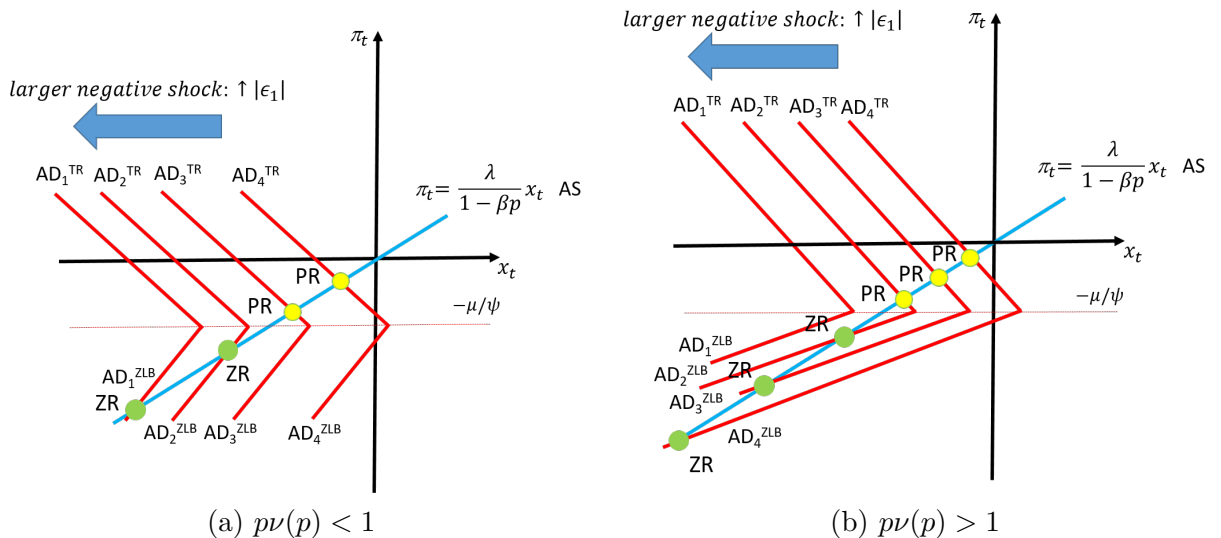Figure 1: Incoherence and Income vs. Substitution

(a) $p\nu(p) < 1$ (b) $p\nu(p) > 1$

Figure 1a graphically illustrates the determination of demand for the case $p\nu(p) < 1$. It can be seen that a solution exists for any ϵ_1. Figure 1b graphically illustrates equilibrium determination when $p\nu(p) > 1$. It is apparent that two solutions exist if ϵ_1 is small, but no solution if ϵ_1 is large in magnitude. In this case, the model is generally incoherent, while, if we impose support restrictions, i.e., $\epsilon_1 > \bar{\epsilon}_{REE}$, the model is incomplete. The issue of incompleteness will be tackled in Section 4.[8]

How should we interpret this restriction on p and ϵ_1? Following Bilbiie (forth.), there are two effects of the demand shock, ϵ_1, when interest rates are pegged at the zero level. First, a larger demand shock (i.e. more negative value of ϵ_1) raises real interest rates given a fixed nominal rate, and this induces households to save more. This intertemporal substitution effect should put downward pressure on inflation and output. At the same time, $\nu(p) > 1$ implies strong income effects at the ZLB; current income, x_t, responds by *more* than proportionally to an increase in expected future output, $E_t x_{t+1}$. For high values of p, an exogenous increase in real interest rates (via lower ϵ_1) raises demand and inflation through this second income effect. In the case where $p\nu(p) > 1$ the income effect dominates the substitution effect, and the negative demand shock has the counter-intuitive effect of raising inflation at the ZLB, while lowering inflation away from the ZLB (see the green and yellow dots respectively in Figure 1b) . In this scenario, we need to make sure that ϵ_1 is not *too* negative. On the other hand, if $p\nu(p) < 1$ then intertemporal substitution effects dominate and more negative ϵ_1 leads to more negative inflation and output, which in turn ensures that a solution with binding ZLB always exists.

In sum, we can discuss the problem of incoherence in our model in terms of income and substitution effects. RE implies that agents are very forward-looking, which in turn can imply a scenario where income effects dominate substitution effects. Tight restrictions on persistence parameter, p, are necessary to avoid this scenario, while restrictions on ϵ_1 are essential to ensure equilibrium when income effects are strong. Much of the rest of this paper investigates whether deviations from RE ensure that these substitution effects dominate income effects when $p\nu(p) > 1$, thus opening up the possibility that non-rational solutions exist when rational solutions may not.

3.2 Coherence without Rationality

We now turn to the question of what happens if no REE exists. Specifically, we investigate the possibility of the existence of non-rational equilibria. First, we look at the case of imperfect knowledge as in Definition 2. Agents with imperfect knowledge are assumed to

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[8]In fact two or four solutions exist in the two cases, respectively, depending on whether one assumes the economy returns to the zero inflation steady – as in Figures 1a and 1b – or one assumes the economy goes to the permanent liquidity trap steady state – not depicted in Figures 1a and 1b. Moreover, the Figures visualize that the condition $p\nu(p) \lesseqgtr 1$ relates to the relative slope of the AS and the AD curve under ZLB. See AM.

recursively estimate simple subjective forecasting models in the spirit of the adaptive learning literature. We assess existence of *temporary equilibria* when agents are learning. Then, we ask if there exists an adaptive learning process that could generate an equilibrium where agents expectations are confirmed. We show that a self-confirming RPE may emerge as the outcome of an adaptive learning process where agents use an under-parameterized forecasting rule and attempt to forecast period-ahead inflation and output using their estimates of the long-run average of both variables. Second, bounded rationality does not need to imply imperfect knowledge, and so it is important to consider what happens when agents are boundedly rational as in Definition 3. It turns out that bounded rationality in the form of discounting ($M, M_f, N < 1$) can imply an even more complete resolution of the problem of incoherence than RPE, provided that the discount factors are exogenously given and do not depend on the magnitude of the shock.

3.2.1 Restricted Perceptions

The model (1)-(3) has a single state variable, ϵ_t, which follows a regime-switching process. Consequently, the REE law of motion for output and inflation is a regime-switching inter-cept – see Definition 4. Rational agents are assumed to know the functional form of the REE solution. However, agents without RE could fail to grasp the structure of the REE, particularly so in the case of incoherence when no such equilibrium exists and therefore it would be impossible to learn/converge to something that does not exist. Consequently, they might try to forecast inflation and output using an under-parameterized forecasting model which omits the state variable, ϵ_t. Agents with these restricted perceptions instead try to forecast the *unconditional* mean of output and inflation:

$$\hat{E}_t Y_{t+j} = Y_t^e = Y_{t-1}^e + t^{-1} \left(Y_{t-k} - Y_{t-1}^e \right), \tag{7}$$

where Y_t^e is the agents' most recent least squares estimate of the unconditional mean of $Y = (x, \pi)'$ using all data available from $t = 0, \ldots, t - k$ where $k = 0$ if agents have current information and $k = 1$ if agents have lagged information and only observe endogenous variables after markets clear. We assume a decreasing gain parameter equal to t^{-1}, but more generally the gain parameter could be a small constant, $g_y \in (0, 1]$ for $y = x, \pi$ ("constant gain learning"), or a mix of constant-gain and decreasing-gain learning as in Marcet and Nicolini (2003). If we substitute (7) into the model then we have the following actual law of motion for Y:

$$Y_t = A_t Y_t^e + B_t, \tag{8}$$

where $A_t = A_Z$ and $B_t = B_{Z,t}$ if the ZLB binds at time-t, and otherwise $A_t = A_P$ and $B_t = B_{P,t}$, and

$$A_P = \begin{pmatrix} \frac{1}{\lambda\sigma\psi+1} & \frac{\sigma-\beta\sigma\psi}{\lambda\sigma\psi+1} \\ \frac{\lambda}{\lambda\sigma\psi+1} & \frac{\beta+\lambda\sigma}{\lambda\sigma\psi+1} \end{pmatrix} \qquad A_Z = \begin{pmatrix} 1 & \sigma \\ \lambda & \beta+\lambda\sigma \end{pmatrix} \qquad (9)$$

$$B_{P,t} = \begin{pmatrix} \frac{\epsilon_t}{1+\lambda\psi\sigma} \\ \frac{\lambda\epsilon_t}{1+\lambda\psi\sigma} \end{pmatrix} \qquad B_{Z,t} = \begin{pmatrix} \epsilon_t+\sigma\mu \\ \lambda\epsilon_t+\lambda\sigma\mu \end{pmatrix}. \qquad (10)$$

Appendix A.2 contains the proof of the following proposition:

Proposition 2 *The model (7)-(8) is coherent and complete if Y_t is not observed contemporaneously (i.e. $k = 1$).*

The preceding proposition makes it clear that learning ensures the existence of a *temporary* equilibrium given any $p, q, \epsilon_1, \epsilon_2$, provided $k = 1$. We consider this to be an inherently significant finding. From a theoretical perspective, it shows that relying on the lagged information assumption, commonly employed in the adaptive learning literature, suffices to solve the coherence problem in a NK model with a ZLB constraint.[9] Intuitively, learning implies that expectations are predetermined, and this simplifies the task of computing the market clearing equilibrium allocation relative to the nontrivial fixed point problem needed to solve for the REE. From an empirical perspective, inflation has been mostly low but stable during and after the Great Recession, contrary to the prediction of deflationary spirals in an RE model. This proposition could provide a possible account of this period, so that inflation is actually determined by a temporary equilibrium, where agent update their beliefs based on under-parameterized forecast rule as data becomes available with a lag.

Though a temporary equilibrium for the economy always exists, learning agents do not have expectations that are necessarily consistent with the data they observe. An equilibrium, instead, is a *self-confirming equilibrium* if the learning agents' subjective inflation and output forecasts coincide the true unconditional means of inflation and output, that is if:

$$\hat{E}_t Y_{t+j} = E(Y) = \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1,$$

where $Y = (x, y)'$, $\hat{\mathbf{Y}}_j$ is Y_t when $\epsilon_t = \epsilon_j$ and $\bar{q} = Pr(\epsilon_t = \epsilon_2) = (1-p)/(2-p-q)$. If the agents form conditional forecasts using the unconditional mean of inflation and output (i.e.

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[9]If $k = 0$ then a temporary equilibrium can fail to exist for small values of t with decreasing gain, or sufficiently large constant gain parameters. Therefore, under contemporaneous information we need to restrict the magnitude of the gain parameter to get a solution. Evans and McGough (2018b) documents that constant gain learning models with contemporaneous information can lead to unreasonable predictions when interest rates are pegged. Proposition 2 is a complementary result that favors the lagged information assumption.

if $\hat{E}_t Y_{t+j} = E(Y)$) then agents' beliefs about the long-run averages of inflation and output are true and self-confirming only if $\hat{\mathbf{Y}}_j$ solves (1)-(3) given $\hat{E}_t Y_{t+j} = E(Y) = \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$.

Definition 5 *Restricted perceptions equilibrium (RPE).* $\hat{\mathbf{Y}} = (\hat{\mathbf{Y}}'_1, \hat{\mathbf{Y}}'_2)'$ *is a restricted perceptions equilibrium if and only if (i)* $\hat{\mathbf{Y}}_j$ *solves (1)-(3) given* $E_t Y_{t+1} = \bar{\mathbf{Y}} := \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1$ *and* $\epsilon_t = \epsilon_j$ *for* $j = 1, 2$; *and (ii)* $E(Y_t) = \bar{\mathbf{Y}}$.[10]

There are four possible RPE of (1)-(3) indexed by $i = PP, ZP, PZ, ZZ$, which are analogous to the REE discussed earlier. Notice that the *actual* law of motion for inflation and output in the RPE is still a regime-switching process. A sufficiently attentive learning agent might be expected to notice that their forecasting model is misspecified in an RPE and consequently, we might question whether this equilibrium concept is "reasonable." A later section discusses issues related to the plausibility of the RPE concept in the context of learning in detail, but in any case, it is worth stressing now that the simple RPE concept put forth in this section is the natural RPE concept for this model. In an RPE, agents have "restricted perceptions" in the sense that they omit key fundamental state variables from their forecasting models, that is, they use an under-parameterized forecast rule. In our simple model, ϵ_t is the only state variable. Consequently, the natural under-parameterized forecast rule for this model omits ϵ_t as (7) does. This RPE concept also makes the analysis tractable, leading to the following useful result.

Figure 2: Restricted Perceptions Equilibrium
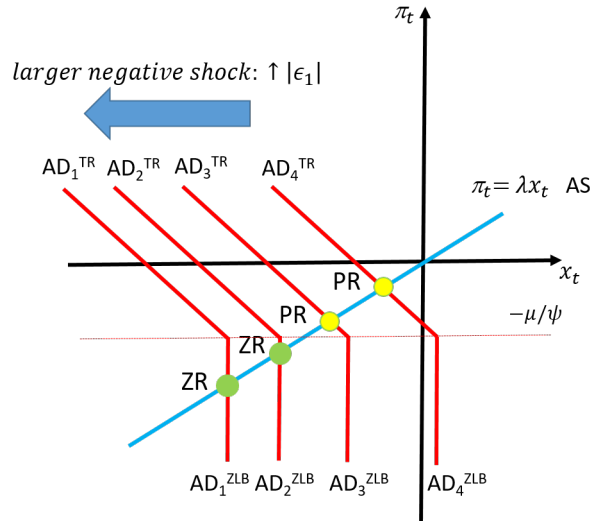
Proposition 3 *Consider (1)-(3) and suppose* $M = M_f = N = 1$, $\epsilon_2 \geq 0$. *Then:*

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[10]See Evans and Honkapohja (2001, sec. 3.6 and 13.1) and Branch (2006) for a thorough discussion of the RPE concept.

i. *An RPE exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{RPE}$, where $\bar{\epsilon}_{RPE}$ depends on the model's parameters, see Equation (A11) in Appendix A.3, and satisfies $\bar{\epsilon}_{RPE} = -\infty$ if $q = 1$.*

ii. *$\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if and only if $p + q \geq 1$.*

Proposition 3 is one of the main results of this paper. It tells us that models with persistent shocks (i.e. $p + q > 1$) admit non-rational equilibria but *not* rational equilibria if $\epsilon_1 \in (\bar{\epsilon}_{RPE}, \bar{\epsilon}_{REE})$.[11] Thus we can gain traction in an otherwise incoherent model of the ZLB by assuming restricted perceptions.

As in the case of REE, it is useful to study RPE when $q = 1$ and $\epsilon_2 = 0$ to develop intuition, see Figure 2. In this case, we have $\bar{q} = 1$ and so the RPE forecast is simply equal to one of the two non-stochastic steady states of the model. Substituting the forecast consistent with the economy reverting to the zero inflation steady state into the model – so $\hat{E}_t x_{t+1} = \hat{E}_t \pi_{t+1} = 0$ in (1)-(3) – and solving for equilibrium output in the temporary state with $\epsilon_t = \epsilon_1$ gives: $x_t = \sigma\mu + \epsilon_1$, assuming the ZLB binds. Thus, effectively the perceived p is equal zero and the slope of the aggregate demand curve becomes vertical in the temporary state under a ZLB. It follows that an RPE exists for *any* p and ϵ_1. No support restrictions for the shock distribution are needed. Restricted perceptions ensures that income effects of raising real rates do not dominate substitution effects, and thus equilibrium is ensured for any assumptions about p and ϵ_1, in accordance with Proposition 3.

3.2.2 Bounded Rationality

Assuming bounded rationality in the form of discounting $(M, M_f, N < 1)$ yields the following proposition that illustrates how deviations from RE ameliorate incoherence concerns, as in Proposition 3.

Proposition 4 *Consider (1)-(3) and suppose $min\{M, M_f, N\} < 1$ and $\epsilon_2 \geq 0$. Then:*

i. *A BRE exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{BR}$, for some constant $\bar{\epsilon}_{BR}$ that depends on the model's parameters, see Equation (A14) in Appendix A.4.*

ii. *If $(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0$ then $\bar{\epsilon}_{BR} = -\infty$.*

Again, we can understand the coherence result in terms of the income and substitution effect of shocks that raises real interest rates at the ZLB. Assume $q = 1$ and $\epsilon_2 = 0$. The BRE value of output in the temporary state binding ZLB is given by:

$$x_t = \nu^{BR}(p)E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1 - M_f\beta p}x_t, -\mu\} + \epsilon_1, \qquad (11)$$

$$\nu^{BR}(p) = \left(M + N\frac{\lambda\sigma}{1 - \beta M_f p}\right).$$

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[11] If $p + q = 1$, then there is no distinction between the REE and RPE. Therefore, we assume $p + q \neq 1$ in practice.

In this bounded rationality model therefore, output at the ZLB is given by

$$x_t = \frac{1}{1 - p\nu^{BR}(p)}(\sigma\mu + \epsilon_1).$$ (12)

Clearly, substitution effects dominate income effects if and only if $p\nu^{BR}(p) < 1$, similar to the RE case. However, unlike the RE case, we have $\nu^{BR}(p) < 1$ if and only if

$$(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0,$$

which is the condition in Proposition 4. Therefore, myopia can ensure that substitution effects dominate income effects for any p (i.e. existence of a MSV solution for any p and ϵ_1).

Not only does $(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0$ ensure coherence in the case of bounded rationality, it also ensures existence of a unique BRE ("completeness").

Proposition 5 *Consider the model given by (1)-(3) and assume $\psi > 1$. A unique bounded rationality equilibrium (BRE) exists for any p, q, ϵ_1 and $\epsilon_2 \geq 0$ if and only if $(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0$. Further, there exist $\epsilon^{PP,BR}$ and $\epsilon^{ZP,BR}$ such that $\epsilon^{PP,BR} > \epsilon^{ZP,BR}$ and*

  i. The PP solution is the unique BRE if and only $\epsilon_1 > \epsilon^{PP,BR}$.

  ii. The ZP solution is the unique BRE if and only if $\epsilon^{PP,BR} \geq \epsilon_1 > \epsilon^{ZP,BR}$.

  iii. The ZZ solution is the unique BRE if and only if $\epsilon_1 \leq \epsilon^{ZP,BR}$.

The condition $(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0$ completely mitigates concerns about incoherence and incompleteness, but the condition requires a rather high degree of discounting in the Euler and Phillips curve equations. As it turns out, the condition is satisfied by Gabaix's preferred calibration: $M = 0.85$, $M_f = 0.8$, $N = 1$, $\beta = 0.99$, $\lambda = 0.11$, $\sigma = 0.2$. For that calibration, we have:

$$(M - 1)(1 - M_f\beta) + \lambda\sigma N = -0.0092 < 0.$$

On the other hand, it is not satisfied for the calibration in McKay et al. (2016a): $M = 0.97$, $M_f = N = 1$, $\beta = 0.99$, $\lambda = 0.02$, $\sigma = 0.375$. That calibration yields:

$$(M - 1)(1 - M_f\beta) + \lambda\sigma N = 0.0072 > 0.$$

Thus bounded rationality offers a full solution of the problems of incoherence and incompleteness for some, but not all, calibrations featured in the literature.

17

3.2.3 BRE, RPE and Coherence

Bounded rationality and imperfect knowledge constitute two distinct departures from RE that are widely discussed in the literature, and they both mitigate concerns about coherence. In this regard, several points are worth considering.

First, bounded rationality might seem to provide a more robust resolution of the problem with respect to imperfect knowledge, as coherence can be ensured for any assumption about p, q and ϵ_t if M, M_f, N are sufficiently small. However, this need not be the case if prices are relatively flexible or if agents choose their discount factors optimally as in Moberly (2022).

To illustrate the importance of price rigidity, Figure 3 depicts different combinations of values of the negative shock, ϵ_1, and of the bounded rationality discount factor, M, that yield coherence in the REE, RPE and BRE cases. The blue line and red line depict $\bar{\epsilon}_{REE}$ and $\bar{\epsilon}_{RPE}$, respectively, and the black line depicts $\bar{\epsilon}_{BRE}$ for different values of ϵ_1 and $M = M_f$. Panels (a), (b) and (c) shows that the difference between $\bar{\epsilon}_{REE}$, $\bar{\epsilon}_{RPE}$, and $\bar{\epsilon}_{BR}$ can be substantial. Panel (a) shows that larger values of M can rule out existence of BRE in cases where an RPE exists. Panel (b) shows that the same result holds even if the expected low demand state duration is calibrated to match the duration of the 2008-2015 U.S. ZLB episode (i.e. $p = 0.965$ implies an expected low state duration of 28 quarters). However, if $M < 0.86$ in the calibrated model then $(M-1)(1-M_f\beta) + \lambda\sigma N < 0$ and $\bar{\epsilon}_{BRE} = -\infty$. Panel (c) reveals that in addition to small M, a high degree of price stickiness (small λ) is necessary for the BRE approach to provide a fuller solution of the incoherence problem than the RPE concept. For high values of λ even heavy cognitive discounting in the Euler equation and Phillips curve will not resolve the problem of incoherence.[12] The so-called "curse of flexibility" is therefore a much more pronounced problem for both REE and BRE than for RPE. This has an important implication because both the theoretical contributions on state-dependent models and empirical evidence on the time-variation of the frequency of price setting convincingly illustrate that the degree of price stickiness might be changing with the economic conditions. Hence, one might expect that in deep recessions where the ZLB is binding persistently, prices should be more flexible and thus λ should be high, making less robust the solution provided by BRE.

BRE also may not exist if agents are assumed to choose their discount factors optimally. So far, as in most applications in the literature of the bounded rationality approach by Gabaix (2020), we kept fixed the cognitive parameters M, M_f, N. However, the degree of attention of agents should be endogenous, and agents might pay more attention when the economy is subject to large shocks, as in deep recessions where the ZLB is binding persistently. Appendix A.6 employs the approach developed by Moberly (2022) to endogenize the degree

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[12]For any M, Mf, N, there is always a large enough value of the product $\lambda\sigma$ to ensure that $(M-1)(1-M_f\beta) + \lambda\sigma N > 0$. Thus, price rigidity and the intertemporal elasticity of substitution play a key role in the existence of BRE.

of attention in the Gabaix (2020) model. In Moberly (2022), firms and households face a cost of paying attention, as in Gabaix (2020), and they choose discount factors, $M_{f,\epsilon_t}, M_{\epsilon_t}$ in order to balance the loss of not paying attention with the cost of paying attention. Appendix A.6 shows that in this case the shock must be bounded for a solution to exist. Intuitively, it is optimal to pay full attention ($M_{f,\epsilon_t} = M_{\epsilon_t} = 1$) when the shock ϵ_1 is sufficiently large in magnitude. However, a solution does not exist when the shock is large and discount factors are high (see Proposition 4). Appendix A.6 details this important caveat, showing that whether bounded rationality solves the problem of incoherence or not hinges on whether discount factors are predetermined or fixed.

Figure 3: Region of Coherence of the REE, RPE, and of the BRE

(a) (b) $p = 0.965$ (c) $\lambda = 0.2$
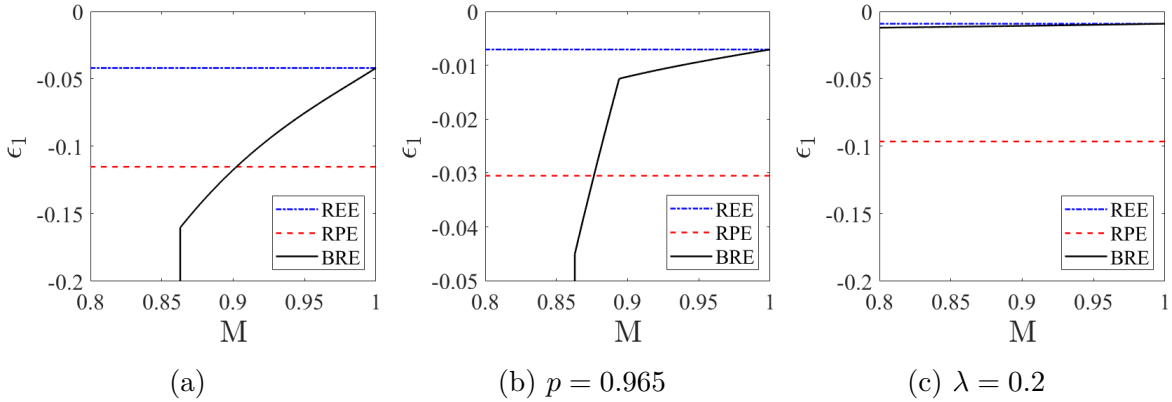
Note: The area above the blue (red) curve depicts values of ϵ_1 for which at least one REE (RPE) exists. The area above the black curve depicts values of ϵ_1 and $M = M_f$ for which at least one BRE exists. Other parameter values: $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $q = 0.98$, $p = 0.85$, $N = 1$, $\epsilon_2 = 0.01$.

Second, the above results cast doubt on the fact that BRE could provide a robust solution to the coherence problem and strengthen the case to consider also the alternative deviation from rationality, that is, imperfect knowledge/adaptive learning. However, it is important to note that the two deviations are not mutually exclusive, and some recent papers have combined imperfect knowledge with myopia or versions of bounded rationality. For example, Hajdini (2022) studies the expectations of myopic agents who have misspecified forecasting models; Meggiorini and Milani (2021) estimates a model that combines adaptive learning and myopia; and Audzei and Slobodyan (2022) derives restricted perceptions equilibrium in an environment that combines adaptive learning and Gabaix's sparse rationality.[13] Similarly, it is possible to combine the two deviations from RE in our model.

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[13]Hajdini (2022) shows that a model with myopic agents with misspecified forecasts accounts for some important empirical facts about inflation forecasts.

Definition 6 *Agents have **bounded rationality and imperfect knowledge** if $\hat{E} \neq E$; $\max\{M, M_f, N\} < 1$ in the NK model given by Equations (1)-(3).*

The analysis in Appendix A.7.1 shows that an environment with boundedly rational agents who have imperfect knowledge could admit a bounded rationality RPE.

Definition 7 ***Bounded rationality restricted perceptions equilibrium (BR-RPE)***. $\hat{\mathbf{Y}} = (\hat{\mathbf{Y}}_1', \hat{\mathbf{Y}}_2')'$ *is a restricted perceptions equilibrium if and only if (i) $\hat{\mathbf{Y}}_j$ solves (1)-(3) given M, M_f, N, $E_t Y_{t+1} = \bar{\mathbf{Y}} := \bar{q}\hat{\mathbf{Y}}_2 + (1 - \bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$; and (ii) $E(Y_t) = \bar{\mathbf{Y}}$.*

There are four possible BR-RPE of (1)-(3) indexed by $i = PP, ZP, PZ, ZZ$, which are analogous to the BRE and RPE discussed earlier. Suitable restrictions on the model ensure existence of BR-RPE.

Proposition 6 *Consider (1)-(3) and suppose $min\{M, M_f, N\} < 1$ and $\epsilon_2 \geq 0$. Then:*

i. A BR-RPE exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{BR,RPE}$, for some constant $\bar{\epsilon}_{BR,RPE}$ that depends on the model's parameters, see Equation (A19) in Appendix A.7.1.

ii. If $(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0$, then $\bar{\epsilon}_{BR,RPE} = -\infty$.

iii. If $(M - 1)(1 - M_f\beta) + \lambda\sigma N \geq 0$ and $p + q \geq 1$ or if $(M - 1)(1 - M_f\beta) + \lambda\sigma N < 0$, then $\bar{\epsilon}_{BR} \geq \bar{\epsilon}_{BR,RPE}$.

The proposition suggests that an environment populated by agents with imperfect knowledge and bounded rationality is especially likely to admit a self-confirming equilibrium. More precisely, the condition for BR-RPE existence is weaker than the condition for BRE existence when the shocks are persistent ($p + q > 1$). Thus, the two deviations from RE are not redundant, and combining them leads to a greater resolution of the incoherence problem than the assumption of bounded rationality by itself, or imperfect knowledge by itself.

Finally, it is well known that bounded rationality can attenuate the so-called "forward guidance puzzle" which is the counter-intuitive prediction that the macroeconomic effects of a promise to cut the interest rate in some future period, T, are strictly increasing in T. Theorem 1 in Appendix A.8 demonstrates that the exactly same condition in Proposition 4.ii, that ensures coherence/completeness in the occasionally-binding constraint framework, also rules out the forward guidance puzzle. Moreover, Propositions 12 and 13 in Appendix A.8 show that the forward guidance puzzle is also absent under imperfect knowledge with adaptive learning. Note that the forward guidance problem is a very different problem from the coherence problem highlighted in this section. First, forward-guidance is generated by a peg of the interest rate, while a peg would not be an issue for coherence, i.e., for the existence of an equilibrium. Second, forward guidance is often modelled as a fixed interest

rate for a known duration (and a known duration of the negative deflationary shock) and then the policy would revert to a standard Taylor rule. Again, if the duration of the shock and of the peg is known, there is no issue of incoherence. Indeed, the model of forward guidance used in Gabaix (2020) and in Appendix A.8 are not susceptible to the problem of incoherence.[14] Thus, both deviations from RE help resolve various puzzles and paradoxes of the New Keynesian ZLB, in addition to resolving the problem of incoherence.

4   Learning to solve incompleteness: multiplicity of (MSV) solutions

We just saw that a BRE can ensure coherence and completeness with sufficient discounting, without any restrictions on the support of the shock. What about completeness in the REE and RPE cases? The coherence condition guarantees existence, but generally that implies a multiplicity of admissible MSV solutions in the case of RE (e.g., Ascari and Mavroeidis, 2022). Incompleteness is by itself a problem which can only be solved using some criterion for selecting an equilibrium. Here we investigate whether learning can provide any guidance, that is, whether the "E-stability" criterion can select an equilibrium of the model as the outcome of an adaptive learning process.

4.1   Learning the REE

In order to define when an REE is E-stable, we first need to be precise about what it means for agents to be learning an REE. As in Section 2, adaptive learning agents have imperfect knowledge and cannot compute an equilibrium analytically. However, these agents make use of a subjective forecasting model or "perceived law of motion" (PLM) when making consumption, labor, savings and pricing decisions consistent with (1)-(2).[15] If the learning agents choose a PLM that is also consistent with how expectations are formed in an REE, then it is possible for learning agents to "learn" an REE if their beliefs about the PLM converge to RE, as beliefs are updated recursively using some statistical scheme for estimating the coefficients of the PLM and observable macro data.

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[14]See also Eusepi et al. (2021), Cole (2021), and Gibbs and McClung (forth.) for more on forward guidance and adaptive learning considerations.

[15]Throughout this paper we restrict our attention to the "Euler equation" approach in which adaptive learning agents are assumed to treat the RE decision rules, (1)-(2), as the decision rules given subjective forecasts. However, under this assumption, agents are not making optimal decision *given* non-rational expectations, as demonstrated by Preston (2005). An alternative approach which accounts for the true optimal consumption and pricing decisions under non-rational expectations is the "infinite horizon learning" approach advanced by Preston (2005) and others. Preliminary results included in Appendix A.12 show that identical RPE existence results can obtain under Euler equation and infinite horizon learning. We leave the full topic of RPE existence and E-stability under infinite horizon learning for future research.

Recall from Section 3.1 that our model admits four possible REE in which output and inflation follow a 2-state process, which are indexed by superscript i to $\mathbf{Y}$, i.e. $\mathbf{Y}^i$ where $i = PP, ZP, PZ, ZZ$. Agents could conceivably learn one of these REE if their PLM for output and inflation is a 2-state process which is estimated recursively using least squares. Consider the following model of learning, in which agents' PLM is a 2-state process for inflation and output, like the REE, and beliefs about the state-contingent means are updated recursively using least squares:

$$Y_{j,t}^e = Y_{j,t-1}^e + t^{-1}\mathcal{I}_{j,t-1}\nu_{j,t-1}^{-1}\left(Y_{t-1} - Y_{j,t-1}^e\right), \tag{13}$$

$$\nu_{j,t} = \nu_{j,t-1} + t^{-1}\left(\mathcal{I}_{j,t-1} - \nu_{j,t-1}\right), \tag{14}$$

$$\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y_{1,t}^e + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y_{2,t}^e, \tag{15}$$

where $j = 1, 2$, $k\nu_{j,k}$ is the number of periods that $\epsilon_t = \epsilon_j$ up until time k, and $\mathcal{I}_{j,t} = 1$ if $\epsilon_t = \epsilon_j$ and $\mathcal{I}_{j,t} = 0$ otherwise (i.e. $\mathcal{I}_{j,t} = 1$ is the indicator function for state j). $Y_{j,t}^e$ is the agents' most recent estimate of the state-contingent average of Y_t when $\epsilon_t = \epsilon_j$. According to equation (13), agents revise their beliefs about the state-contingent average of Y in state j (i.e. $Y_{j,t}^e$) in the direction of their time-$t - 1$ forecast error only if $\epsilon_{t-1} = \epsilon_j$ (otherwise, $Y_{j,t}^e = Y_{j,t-1}^e$). Equation (15) then gives agents' time-t forecast of period-ahead inflation and forecast. It is assumed that agents observe ϵ_t when forecasting at time-t and also that $Pr(\epsilon_{t+1}|\epsilon_t)$ coincides with the actual transition probabilities – e.g. agents know $Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t = \epsilon_1) = p$ and $Pr(\epsilon_{t+1} = \epsilon_2|\epsilon_t = \epsilon_2) = q$. After agents form time-t expectations, we obtain the time-t market-clearing equilibrium, Y_t by substituting equation (15) into the model (1)-(3). The process repeats itself at time $t + 1$ and so on.[16]

We are interested in knowing if $(Y_{1,t}^e, Y_{2,t}^e) \to (\mathbf{Y}_1^i, \mathbf{Y}_2^i)$ for some REE i as time goes on $(t \to \infty)$ and agents' expectations evolve according to (13)-(15). We say that REE i is "stable under learning" if $(Y_{1,t}^e, Y_{2,t}^e) \to (\mathbf{Y}_1^i, \mathbf{Y}_2^i)$ almost surely. When might this convergence of subjective beliefs to RE occur? To make this question tractable, assume that $Y_t^e = (Y_{1,t}^{e'}, Y_{2,t}^{e'})'$ is sufficiently near REE i, such that the ZLB binds under adaptive learning if and only if the ZLB would bind in REE i. This implies the following actual law of motion for Y:

$$Y_t = A_t^i\left(Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y_{1,t}^e + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y_{2,t}^e\right) + B_t^i, \tag{16}$$

for $i \in \{PP, PZ, ZP, ZZ\}$, where $A_t^{PP} = A_P$ and $B_t^{PP} = B_{P,t}$ for all t; $A_t^{ZZ} = A_Z$ and $B_t^{ZZ} = B_{Z,t}$ for all t; $A_t^{ZP} = A_P$ and $B_t^{ZP} = B_{P,t}$ if $\epsilon_t = \epsilon_2$ and $A_t^{ZP} = A_Z$ and $B_t^{ZP} = B_{Z,t}$

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[16]Closely related learning algorithms are used by Woodford (1990), Evans and Honkapohja (1994) and (Evans and Honkapohja, 2001, p.305-308) to study the E-stability of sunspot equilibria involving discrete-valued shocks, and by Evans and Honkapohja (1998) to study learnability of fundamental equilibria with exogenous shocks following a finite state Markov chain. We arrive at identical E-stability results if we alternatively assume least squares estimation of a PLM of the form: $Y_t^e = \hat{a} + \hat{b}\mathcal{I}_t$ where $\mathcal{I}_t = 1$ if $\epsilon_t = \epsilon_2$ and 0 otherwise.

otherwise; $A_t^{PZ} = A_P$ and $B_t^{PZ} = B_{P,t}$ if $\epsilon_t = \epsilon_1$ and $A_t^{PZ} = A_Z$ and $B_t^{PZ} = B_{Z,t}$ otherwise, where the matrices A_P, A_Z and the vectors $B_{P,t}, B_{Z,t}$ are defined in (9) and (10), respectively.

Given beliefs that are local to RE beliefs, we assess the learnability of equilibrium using the E-stability principle. An REE i is said to be E-stable if it is a locally fixed point of the ordinary differential equation (ODE):

$$\frac{\partial \tilde{Y}^e}{\partial \tau} = H^i(\tilde{Y}^e), \qquad \text{where} \qquad H^i(\tilde{Y}^e) = \begin{pmatrix} Y_1^i(Y_1^e, Y_2^e) \\ Y_2^i(Y_1^e, Y_2^e) \end{pmatrix} - \begin{pmatrix} Y_1^e \\ Y_2^e \end{pmatrix} \tag{17}$$

and $Y_j^i(Y_1^e, Y_2^e)$ is the value of Y when $\epsilon_t = \epsilon_j$ as a function of expectations, $\tilde{Y}^e = (Y_1^{e\prime}, Y_2^{e\prime})'$. The relevant Jacobian for assessing the E-stability of REE i is: $DT_{Y^i} = \frac{\partial H^i(\tilde{Y}^e)}{\partial \tilde{Y}^e}|_{\tilde{Y}^e = \mathbf{Y}^i}$. An REE i is E-stable if the eigenvalues of DT_{Y^i} have negative real parts, see Evans and Honkapohja (2001).

There is an intuition for the link between the E-stability condition and stability of beliefs. The ODE (17) is an approximation of the dynamics of Y_t^e near the REE for large t, and it tells us that agents' expectations are revised in the direction of the forecast error, $\bar{Y}^i(Y^e) - Y^e$. If the roots of $DT_{\bar{Y}^i}$ have negative real parts, then agents' expectation about the unconditional mean of inflation and output are also revised in the direction of their REE values.

We note the E-stability conditions applied to the REE of the occasionally binding constraint model are identical to the E-stability conditions applied to a model that features exogenous Markov-switching in the monetary policy stance driven entirely by ϵ_t (e.g., see Branch et al., 2013; McClung, 2020).[17] For example, the E-stability condition associated to the ZP equilibrium of (1)-(3) is the same condition associated to the MSV solution of a model that assumes $i_t = \psi \pi_t$ if $\epsilon_t = \epsilon_2$ and $i_t = -\mu$ if $\epsilon_t = \epsilon_1$ regardless of whether the ZLB binds.

Applying the E-stability to the model at hand leads us to the conclusion that only one REE has the property of being E-stable (see Appendix A.9 for the proof).

Proposition 7 *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. Then:*

i. If $\epsilon_1 > \bar{\epsilon}_{REE}$, at most one E-stable REE exists.

ii. The E-stable REE is either the PP REE or the ZP REE.

Proposition 7 somewhat extends insights from Christiano et al. (2018) to models with recurring low demand states (i.e. $q < 1$). Thus Proposition 7 can be applied to study an economy such as the U.S. economy, which has visited the ZLB twice since 2007, following two distinct negative shocks to the economy. The result in Proposition 7 makes it clear that while multiple solutions exist, only one of them can be understood as the outcome of an adaptive learning process. Hence, incompleteness is resolved by E-stability.

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[17]Mertens and Ravn (2014) also derive E-stability conditions for an equilibrium of a simple New Keynesian model with ZLB constraint, assuming a 2-state discrete sunspot shock with an absorbing regime.

Pinning down the forecasting model. Proposition 7 assumes that agents believe that output and inflation follow a 2-state process, consistent with REE. However, the REE law of motion can be represented in a variety of different ways. For instance, consider the following perceived laws of motion for inflation and output:

$$Y_t^e = a_{\epsilon_{t-k}}, \tag{18}$$

$$Y_t^e = a_{\epsilon_{t-k}} + b\epsilon_{t-k}, \tag{19}$$

$$Y_t^e = a + b_{\epsilon_{t-k}}\epsilon_{t-k}, \tag{20}$$

$$Y_t^e = a_{\epsilon_{t-k}} + b_{\epsilon_{t-k}}\epsilon_{t-k}, \tag{21}$$

$$Y_t^e = a + b\epsilon_{t-k}, \tag{22}$$

$$z_t^e = a_z + b_z z_{t-1} \tag{23}$$

where $z \in \{\pi, x\}$, $k = 0, 1$ and $a_{\epsilon_{t-k}}$, $b_{\epsilon_{t-k}}$ may assume different values depending on ϵ_{t-k}. Again, Y_t^e denotes the subjective forecast of Y_t implied by the forecasting model.

If learning agents instead had one of the PLMs (18)-(23) and estimated the parameters of those models recursively, e.g. using least squares, would they eventually have self-confirming views about inflation and output? In other words, would the data confirm their belief that Y_t follows one of the processes (18)-(23)? If agents observe ϵ_t and Y_t when forecasting at time t, then beliefs formed under PLMs of the form (18)-(23) can only become self-confirming if an REE exists. Hence, we refer to (18)-(23) as "REE-consistent beliefs".

Proposition 8 *Suppose agents condition time-t forecasts on current (time-t) variables. Then REE-consistent beliefs (18)-(23) can only be self-confirming if an REE exists.*

Proposition 8 (proof in Appendix A.10) makes it apparent that agents including the demand shock, ϵ_t, in their (piecewise) linear forecasting model (or Y_t in the case of (23)) cannot develop self-confirming views about the economy if an REE does not exist (incoherence). This result has implications for how we should think about learning and equilibrium in the case of incoherence. Since none of the above "REE-consistent beliefs", i.e., PLMs consistent with an REE following a 2-state process, could converge to a self-confirming equilibrium whenever the REE does not exist, it means that we should look at different PLMs in case of incoherence, such as an RPE, to which we turn next.

4.2 Learning the RPE

We now turn to the question of learnability of RPE. Propositions 7 and 8 tell us that agents with imperfect knowledge using any of the subjective forecasting models discussed in the previous section will not learn any self-confirming equilibrium whenever an REE does not exist. However, Proposition 3 shows that an RPE can exist even if an REE does not. It

turns out multiple RPE may exist when the restrictions in Proposition 3 hold. Can one or more of these RPE emerge as the outcome an econometric learning process, similar to what we considered in the case of REE? The answer is yes. Here we show that the model may still admit one unique learnable, self-confirming RPE.

First, we must assume agents have a subjective PLM for output and inflation that is consistent with how expectations are formed in an RPE, which is given by equation (7). If we substitute (7) into the model and assume Y_t^e is sufficiently near RPE i then we have the following actual law of motion for Y:

$$Y_t = A_t^i Y_t^e + B_t^i, \tag{24}$$

where $A_t^{PP} = A_P$ and $B_t^{PP} = B_{P,t}$ for all t; $A_t^{ZZ} = A_Z$ and $B_t^{ZZ} = B_{Z,t}$ for all t; $A_t^{ZP} = A_P$ and $B_t^{ZP} = B_{P,t}$ if $\epsilon_t = \epsilon_2$ and $A_t^{ZP} = A_Z$ and $B_t^{ZP} = B_{Z,t}$ otherwise; $A_t^{PZ} = A_P$ and $B_t^{PZ} = B_{P,t}$ if $\epsilon_t = \epsilon_1$ and $A_t^{PZ} = A_Z$ and $B_t^{PZ} = B_{Z,t}$ otherwise.

We say that RPE i is stable under learning if $Y_t^e \to \bar{\mathbf{Y}}^i$ almost surely, where $\bar{\mathbf{Y}}^i$ denotes the unconditional mean of Y_t^i. Analogous to the discussion of E-stability of REE above, we say that RPE i is said to be E-stable if it is a locally stable fixed point of the ODE, $\partial Y^e / \partial \tau = h^i(Y^e)$, where $h^i(Y^e) = \bar{Y}^i(Y^e) - Y^e$, where $\bar{Y}^i(Y^e)$ is the unconditional mean of Y as a function of expectations, Y^e. Formally, E-stability obtains if the eigenvalues of the Jacobian, $DT_{\bar{Y}^i} = \frac{\partial h^i(Y_e)}{\partial Y^e}\big|_{Y^e = \bar{Y}^i}$ have negative real parts. An E-stable RPE is stable under learning if agents estimate Y_t^e using least squares, as in (7), or related estimation routines.[18]

Proposition 9 *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. If $\epsilon_1 > \bar{\epsilon}_{RPE}$, then:*

i. There is a unique E-stable RPE.

*ii. The E-stable RPE is either the PP RPE or the ZP RPE.*

4.3   Is the RPE reasonable?

In an RPE, agents have badly misspecified beliefs. Agents forecast the mean of inflation and output as if they believe those variables are constant or mean-plus-noise, despite the fact that these variables would obviously follow a persistent 2-state Markov chain in an RPE. Why would we consider RPE reasonable? Should agents be expected to detect their mis-specification over time simply by looking at time series data? Few comments are in order.

First, if an REE exists, then we could argue these RPE are implausible. In this case, agents could learn to do better, because there would likely be a learnable REE. But incoherence precludes REE, and as shown in Proposition 8, it implies that agents fail to form

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[18]In Appendix A.7.2, Proposition 11 shows that a unique E-stable BR-RPE exists in the case where agents both are boundedly rational and have imperfect knowledge.

Figure 4: Region of Coherence of the REE and of the RPE
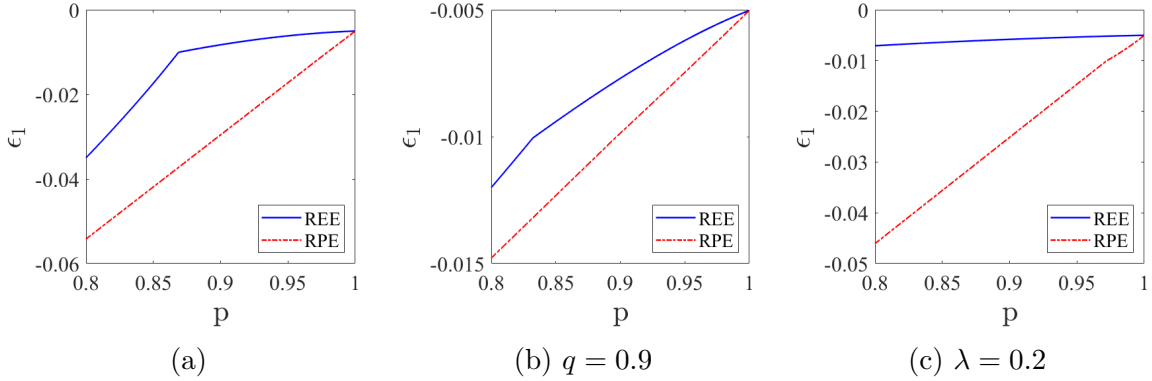
Note: The area above the blue (red) curve depicts values of ϵ_1 and p for which at least one REE (RPE) exists. Other parameter values: $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $q = 0.98$, $\epsilon_2 = 0$.

self-confirming expectations using a variety of different forecasting models that condition on the demand shock or even a lag of the endogenous variables. In the case of incoherence of REE, the RPE is thus a potentially reasonable alternative, because it relaxes the condition for the existence of self-confirming equilibria. Figure 4 visualizes the difference between the combination of values of the negative shock, ϵ_1, and of its persistence, i.e., p, that yields coherence in the REE and in the RPE cases. The area above the blue line and the red line defines the set of pairs (ϵ_1, p) so that at least one REE and RPE exist, respectively. Panel (a) shows that the difference between the region of the parameter space for which there is coherence in the two cases is substantial. In particular, unless the persistence, p, of the negative demand shock falls below 0.87, RE admits an equilibrium only for very small negative shocks. Panel (b) shows that both regions are quite sensitive – they shrink by around a quarter – to the value of the persistence of the other state where $\epsilon_2 = 0$. Finally, panel (c) shows that the region of coherence of REE shrinks quite substantially as prices becomes more flexible, while this is not the case for the RPE. The curse of flexibility is therefore a much more pronounced problem for REE than for RPE, just as Figure 3 (c) shows, which is very intuitive because the curse hinges on the rationality and forward-lookingness of the agents.

The Figure 4 results suggest that a fundamentals-driven RE liquidity trap must be relatively short-lived in the case of an REE compared to the duration of actual liquidity trap events experienced by Japan, the Euro Area and the U.S. In contrast, a fundamentals-driven RPE liquidity trap can be more persistent. Figure 5 depicts the maximum expected duration of the liquidity trap (equal to $(1-p)^{-1}$) that we can generate in a ZP REE or ZP RPE for different combinations of demand shock, ϵ_1. It can be seen that liquidity traps cannot be very persistent in an REE, whereas the RPE liquidity traps can be highly persistent,

26

particularly if q is relatively large as in panel (a).[19] Panel (c) again shows that the curse of flexibility is a more pronounced problem for the REE. The BRE results are not depicted in Figure 5, but Proposition 5 implies that we can generate permanent ZLB events in a BRE for very negative shocks.

Figure 5: Maximum Expected ZLB Duration in a ZP Solution

| (a) | (b) $q = 0.9$ | (c) $\lambda = 0.2$ |
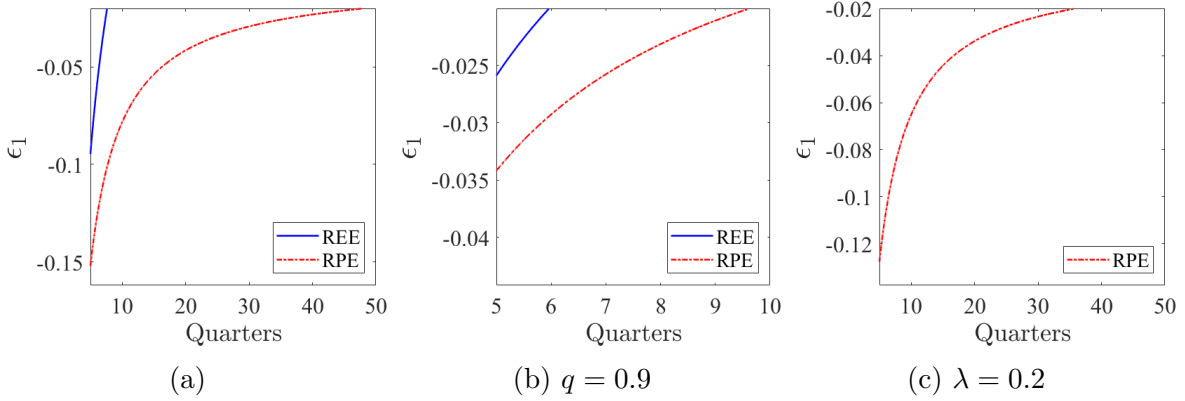
Note: The blue (red) curve depicts the maximum expected duration ZLB $((1 - p)^{-1})$ we can generate for given ϵ_1 in an REE (RPE) ZP solution. The figure only depicts values of ϵ_1 for which an REE ZP or RPE ZP solution exists. Other parameter values: $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $q = 0.98$, $\epsilon_2 = 0.01$.

Second, suppose the model is incoherent under RE, but an E-stable RPE exists and the economy is in it. One could argue that agents inhabiting the RPE would notice that RPE inflation and output follow a 2-state process. Hence, agents would then stop setting one-period ahead inflation and output expectations equal to the long run average of those variables, and start to estimate a 2-state forecasting model in their attempt to learn these dynamics. Our previous propositions already suggest this might be a bad idea. Indeed, Propositions 7 and 8 establish that such beliefs cannot be self-confirming. Can they reach another – not self-confirming – equilibrium? Figure 6 (a) depicts the results from simulating the learning dynamics for the case of MSV-consistent beliefs and also for the case of RPE-consistent beliefs, assuming a small constant gain.[20] It clearly shows that MSV-consistent beliefs are explosive even with very small gain parameter, while, on the contrary, the RPE-consistent beliefs are not. Panel (b) in Figure 6 displays the dynamics of expected inflation (and its cumulative average in red) from which it is evident that RPE expectations remain

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[19]Note that $p = 0.965$ produces an expected liquidity trap duration of around 28 quarters, which is the length of the 2008-2015 ZLB episode in the U.S.

[20]For MSV learning simulation, we initialize the forecast, $Y_{j,1}^e$ to match the state-contingent mean of inflation/output in the RPE when $\epsilon_t = \epsilon_j$. In other words, we assume that agents observe actual endogenous variables in the RPE switching with ϵ_t during periods $t < 1$ and then they decide to make their forecasts consistent with the switching at $t = 1$. We use the same initialization for RPE beliefs.

Figure 6: Simulations when REE doe not exist and an RPE exists

(a) RPE vs MSV beliefs   (b) RPE beliefs
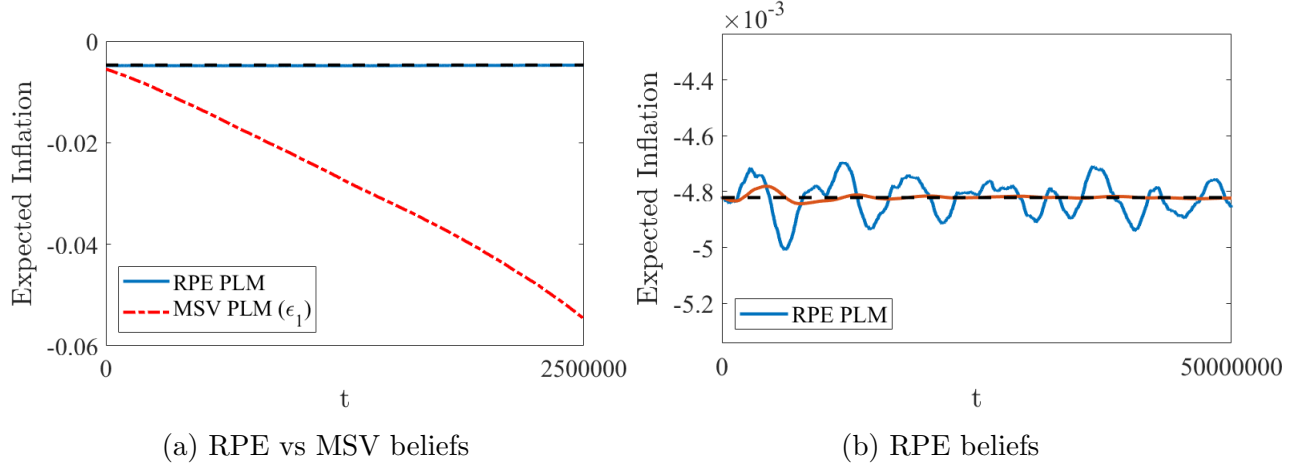
Note: The model is calibrated so that an E-stable RPE ZP solution exists, but no MSV REE exists. The constant gain is small and set to $g_t = 0.00001$ for all t. $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $p = 0.85$, $q = 0.98$, $\epsilon_1 = -0.04$, $\epsilon_2 = 0$.

in some neighborhood around their RPE values.[21] Numerical simulation therefore suggests another reason why the RPE might be a good alternative. If an RPE exists – and an REE does not – and if agents try to learn using the REE PLM, then the economy will derail into deflationary spirals. On the contrary, if agents try to learn the RPE, then expectations remain stable and "centered" on the correct RPE values – provided that the gain parameter is small and initial inflation and output expectations are not too far away from the average inflation and output rate in the RPE.

Third, it is important to recall from Poposition 2 that the assumption of learning by itself ensures coherence and completeness, provided that agents have lagged information. Thus, while tight model restrictions are needed to characterize self-confirming equilibrium, the economy can *always* be in a market-clearing temporary equilibrium.

Of course there could be other non-rational equilibria such as the consistent expectations equilibrium considered by Jorgensen and Lansing (2021), the stochastic consistent expectations equilibria (SCEE) of Hommes and Zhu (2014) or Airaudo and Hajdini (2021). In particular, an SCEE arises if agents use a forecasting model akin to (23) with lagged information about endogenous variables and the agents' beliefs about the mean *and* autocorrelation of inflation and output implied by the forecasting model is confirmed by the observable economic data. In a SCEE, agents' forecasts introduce a lag of inflation and output into

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[21]Moreover, simulations – not reported – also show that RPE-consistent beliefs tend to revert to RPE values even with decreasing gain and when initial beliefs are a small distance from RPE values. Intuitively, the RPE-consistent beliefs could also be explosive (into deflationary spirals) whenever the gain parameter is too large or initial beliefs are very far from the RPE value.

the model, which prevents us from analyzing the existence of SCEE in our model with an occasionally binding constraint.[22] Our numerical analysis indicates that these more sophisticated non-rational equilibria may not exist for some plausible calibrations of the model.[23] Thus, the RPE may even be the best alternative among non-rational equilibria of our model with $M = M_f = N = 1$, but CEE or SCEE existence remains an open question. However, whether or not these alternative non-rational equilibria exist is not relevant for the main result of this paper: rationally incoherent models are non-rationally coherent, i.e., admit non-rational equilibria.

4.4 RPE and Continuous Shocks

To get closed-form solutions for both REE and RPE, we must assume that ϵ_t follows a discrete-valued Markov chain. To the best of our knowledge, no paper provides conditions for existence and uniqueness of RE equilibrium which can be applied to a model similar to our model under the assumption that ϵ_t is both persistent and continuously distributed.[24] However, while it is hard to characterize REE in a model with continuous shocks and an occasionally binding constraint, it is relatively easy to derive RPE.

To illustrate, consider the model (1)-(3) and suppose instead that $\epsilon_t = \rho \epsilon_{t-1} + v_t$ where $\rho \in [0,1)$ and $v_t \sim \mathcal{N}(0, \sigma_v^2)$. In an RPE of this economy, agents' forecasts are given by $\hat{E}_t \pi_{t+1} = a_\pi$, $\hat{E}_t x_{t+1} = \frac{1-\beta}{\lambda} a_\pi$ consistent with the RPE studied in the previous sections. Substituting these expectations into the model gives the following RPE law of motion for inflation:

$$\pi_t = \begin{cases} (1 + \lambda\sigma)a_\pi + \lambda\sigma\mu + \lambda\epsilon_t & \text{if } s_t = 0, \\ \frac{1+\lambda\sigma}{1+\lambda\sigma\psi}a_\pi + \frac{\lambda}{1+\lambda\sigma\psi}\epsilon_t & \text{if } s_t = 1. \end{cases} \tag{25}$$

Let $h(a_\pi)$ denote $E(\pi_t)$ as a function of a_π, and let $\sigma_\epsilon = \sqrt{\frac{\sigma_v^2}{1-\rho^2}}$. Then:

$$h(a_\pi) = Pr\left(s_t = 0\right) E(\pi_t | s_t = 0) + (1 - Pr\left(s_t = 0\right)) E\left(\pi_t | s_t = 1\right), \tag{26}$$

where $s_t = 0$ indicates that the ZLB is binding. To compute RPE, we need to compute $Pr(s_t = 0)$, $E(\pi_t | s_t = 0)$ and $E(\pi_t | s_t = 1)$ as functions of a_π. Let Φ and ϕ denote the standard normal probability distribution function and standard normal probability density function, respectively. Further, define:

$$L(a_\pi) := (\sigma_\epsilon \lambda)^{-1} \left(-\mu/\psi - (1 + \lambda\sigma)a_\pi - \lambda\sigma\mu\right) \tag{27}$$

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[22]See, e.g., AM for a discussion on the difficulty of studying models with occasionally binding constraints and lagged endogenous variables

[23]Results are available on request.

[24]See Mendes (2011) for analytical existence results under the assumption that ϵ_t is a mean-zero, i.i.d process.

It follows that:

$$Pr(s_t = 0) = \Phi(L(a_\pi)),$$

$$E(\pi_t | s_t = 0) = (1 + \lambda\sigma)a_\pi + \lambda\sigma\mu - \frac{\lambda\sigma_\epsilon \phi(L(a_\pi))}{\Phi(L(a_\pi))},$$

$$E(\pi_t | s_t = 1) = \frac{1 + \lambda\sigma}{1 + \lambda\sigma\psi}a_\pi + \frac{\lambda\sigma_\epsilon \phi(L(a_\pi))}{(1 + \lambda\sigma\psi)(1 - \Phi(L(a_\pi)))}.$$

Therefore, we have :

$$h(a_\pi) = \frac{1 + \lambda\sigma}{1 + \lambda\sigma\psi}a_\pi + \Phi(L(a_\pi))\left(\frac{(1 + \lambda\sigma)\lambda\sigma\psi}{1 + \lambda\sigma\psi}a_\pi + \lambda\sigma\mu\right) - \frac{\phi(L(a_\pi))\lambda^2\sigma_\epsilon\sigma\psi}{1 + \lambda\sigma\psi}. \qquad (28)$$

There is an RPE if and only if there exists $\bar{a}_\pi \in \mathbb{R}$ such that $h(\bar{a}_\pi) = \bar{a}_\pi$. One can show there exists a unique maximum of $h(a_\pi) - a_\pi$, denoted a_π^*, and consequently there is either no RPE solution or there are exactly two RPE solutions.[25] A necessary and sufficient condition for existence of the RPE is $h(a_\pi^*) - a_\pi^* \geq 0$. We summarize the result as a proposition.

Proposition 10 *Consider (1)-(3) and suppose that $\epsilon_t = \rho\epsilon_{t-1} + v_t$ where $v_t \sim \mathcal{N}(0, \sigma_v^2)$. Then:*

i. Two RPE exist if and only if $h(a_\pi^*) > a_\pi^*$ where a_π^* is given by

$$a_\pi^* = L^{-1}\left(\Phi^{-1}\left(\frac{\psi - 1}{(1 + \lambda\sigma)\psi}\right)\right).$$

*ii. An RPE does not exist if and only if $h(a_\pi^*) < a_\pi^*$.*

By inspecting (28), one can see that increasing the variance and persistence of the shocks (i.e. increasing σ_v and ρ) or decreasing price rigidity (i.e. increasing λ) reduces $h(a_\pi) - a_\pi$, that needs to be positive for an (actually two) RPE to exist. Consequently, sufficiently high values of σ_v, ρ or λ preclude existence of RPE in the model with continuous, persistent shocks. Figure 7 plots $h(a_\pi) - a_\pi$ for three different values of σ_v, assuming $\rho = 0.8$. It is evident that larger values of σ_v shifts $h(a_\pi) - a_\pi$ down.[26] Notice in the figure that the RPE levels of inflation are always less than the zero inflation steady state level, and hence the numerical RPE we consider display a deflationary bias akin to the deflationary bias studied under RE in Nakata and Schmidt (2019a) or Bianchi et al. (2021). Figure 8 plots $h(a_\pi^*) - a_\pi^*$

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[25]To see this, note that Φ is strictly decreasing in a_π and Φ and L are injective functions and that $h'(a_\pi) - 1 = \frac{\lambda\sigma(1-\psi)}{1+\lambda\sigma\psi} + \frac{\Phi(L(a_\pi))\lambda\sigma\psi(1+\lambda\sigma)}{1+\lambda\sigma\psi}$. Then under the Taylor Principle ($\psi > 1$), there exists a unique maximum, a_π^*, such that $h'(a_\pi^*) - 1 = 0$ and $h'(a_\pi) - 1 > 0$ ($h'(a_\pi) - 1 < 0$) for all $a_\pi < a_\pi^*$ ($a_\pi > a_\pi^*$). For brevity, we abstract from the special case in which $h(a_\pi^*) = a_\pi^*$.

[26]Figures 7 and 8 plot $h(a_\pi^*) - a_\pi^*$ for different calibrations of key parameters. In both figures we use the following benchmark calibration unless otherwise noted: $\beta = 0.99$, $\sigma = 1$, $\psi = 2$, $\lambda = 0.02$, $\rho = 0.8$, $\sigma = 0.1$.

for different values of other key parameters in calibrated models. To interpret the panels in the Figure recall that $h(a_\pi^*) - a_\pi^* > 0$ for the RPE to exist. The figure shows that the RPE is less likely to exist if the shock variance or persistence is high, or if prices are more flexible. Hence, the same insights from the simple 2-state process example carry over to the case of continuous shocks (see Figure 4).

Figure 7: Existence and Multiplicity of RPE with Continuous Shocks
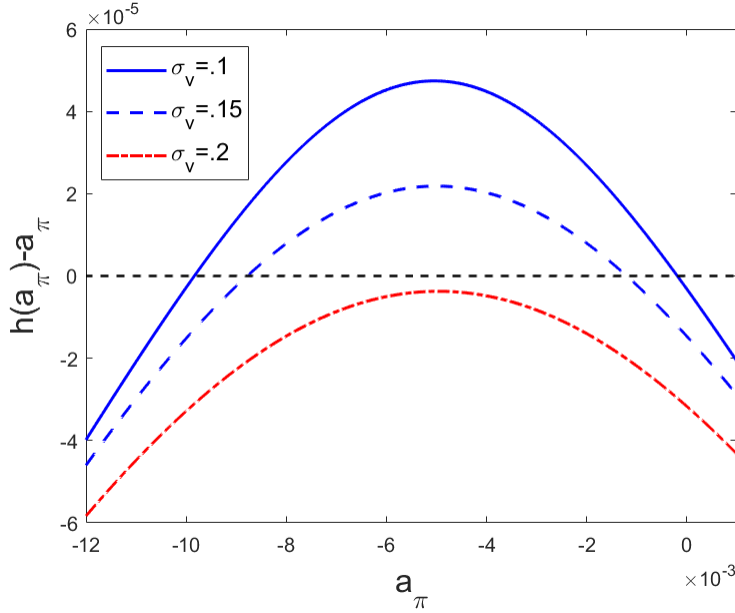
## 5 Variation on a theme: REE with lagged expectations

This section briefly looks at the possibility of the existence of other equilibria, in which agents have imperfect information in the sense that they do not observe the contemporaneous shock. Throughout this paper we stuck to the standard assumption that "rational" agents observe the demand shock contemporaneously (i.e. ϵ_t is included in agents' time-t information set). This would be a natural assumption if for example ϵ_t is a shock to the households' preferences as in Eggertsson and Woodford (2003). However, the assumption that agents observe ϵ_t with a lag (so that ϵ_{t-1}, but not ϵ_t, is included in agents' time-t information set) permits the study of some additional non-rational equilibria which may exist in rationally incoherent models.

To illustrate existence of these additional "lagged expectations equilibria" (LEE), consider the model (1)-(3) and suppose $q = 1$, $\epsilon_2 = 0$. Further suppose that agents believe inflation and output follows the same persistent 2-state Markov chain as the shock (just like rational agents) but instead agents do not know ϵ_t and hence agents attach p^2 probability
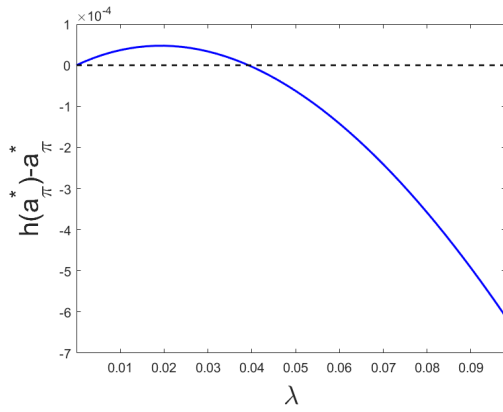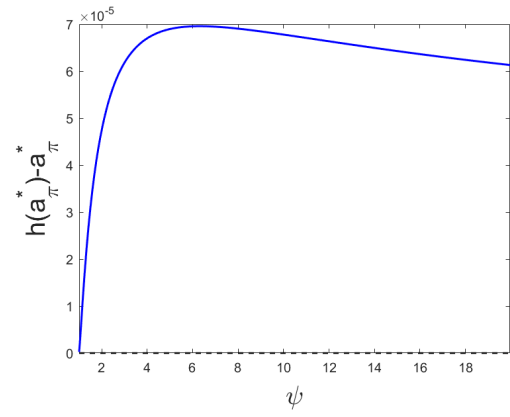
Figure 8: RPE Existence
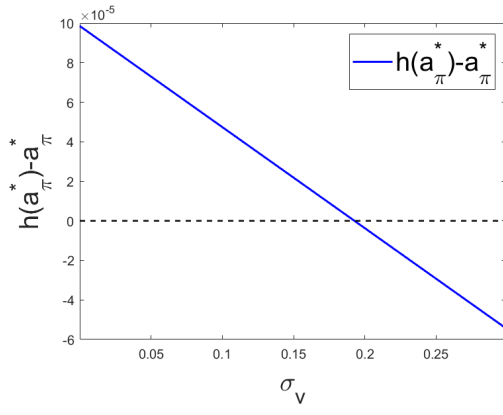
(a) Shock Variance
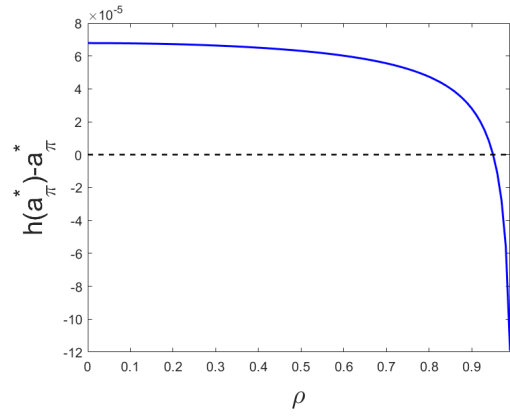
(b) Shock Persistence

(c) Price Flexibility

(d) Activeness of Policy

to the prospect that $\epsilon_{t+1} = \epsilon_1$ when forecasting at time t in the temporary state, instead of attaching p probability to this event as agents with full-information RE would do. Under this assumption about agents' time-t information set, the economy either returns to the steady state with zero inflation or the steady state with zero interest rates after $\epsilon_t = \epsilon_2$. The "temporary state" value of output when $\epsilon_t = \epsilon_1$ (assuming for simplicity that we go back to the zero inflation steady state) is given by:

$$x_t = \nu(p^2)\hat{E}_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-\beta p^2}x_t, -\mu\} + \epsilon_1 \tag{29}$$

$$\text{where} \quad \nu(p^2) = \left(1 + \frac{\lambda\sigma}{1-\beta p^2}\right) > 1,$$

which we obtain by substituting the Phillips curve and Taylor rule into (1). From this equation, it is apparent that for any p, sufficiently low values of ϵ_1 preclude unconstrained interest rates, just as in the case of full information RE. Thus, for a sufficiently large demand shock, output will be given by;

$$x_t = \frac{1}{1-p^2\nu(p^2)}(\sigma\mu + \epsilon_1), \tag{30}$$

if a solution of the model exists at all. We call this solution a lagged expectation equilibrium (LEE). It is a self-confirming equilibrium because agents correctly forecast the conditional mean of output and inflation (e.g. $E(x_t|\epsilon_t = \epsilon_1) = \frac{1}{1-p^2\nu(p^2)}(\sigma\mu+\epsilon_1)$ and $E(x_t|\epsilon_t = \epsilon_2) = 0$).[27] Note that $p^2\nu(p^2) < p\nu(p)$, and therefore if $p^2\nu(p^2) < 1 < p\nu(p)$ we will have a LEE given any ϵ_1, but only an REE if ϵ_1 is sufficiently close to zero. REE existence always implies existence of LEE, but the opposite is not true. This simple exercise reveals that there can be additional deviations from RE, beyond the scope of this paper, which are useful for understanding an incoherent model.

6 Concluding remarks

Standard RE models with an occasionally binding zero lower bound (ZLB) constraint either admit no solutions (incoherence) or multiple solutions (incompleteness). This paper shows that the problem of incompleteness and incoherence hinges on the assumption of RE.

Models with no rational equilibria may admit self-confirming equilibria involving the use of simple mis-specified forecasting models. The main message of the paper from the existence analysis is that when negative shocks are sufficiently large in magnitude or sufficiently persistent, the baseline NK model is incoherent, but can admit RPE or BRE. Completeness and

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[27]In the first period such that $\epsilon_t = \epsilon_2$, we have $x_t \neq 0$. However, $E(x_t|\epsilon_t = \epsilon_2) = E(\pi_t|\epsilon_t = \epsilon_2) = 0$ because state 2 is an absorbing state. Thus, the LEE is a non-rational equilibrium in which agents have self-confirming beliefs about the state-contingent conditional means of endogenous variables.

coherence can be restored if expectations are adaptive or if agents are less forward-looking due to some information or behavioral friction.

In the case of multiple solutions, the E-stability criterion selects an equilibrium. An RPE can exist as a self-confirming equilibrium, even if the underlying model does not admit an REE. Thus, non-rationality of agents' beliefs can save the economy from blowing up into infinite deflationary spirals, while it yields persistent liquidity traps. These results highlight how deviations from RE help us understand persistent liquidity traps in theoretical models and interpret the recent episodes of liquidity traps in Japan, the Euro Area, and the U.S.

We leave room for future work. In particular, we used the RPE and BRE concepts to make our point simple and clear, and consequently we abstracted from other self-confirming equilibria that could emerge under adaptive learning, such as CEE or SCEE. Similarly, we excluded other popular forms of non-rationality from our analysis, such as level-k reasoning, or social memory frictions as in Angeletos and Lian (forth.).

Finally, we put a premium on analytical results and therefore we focused on a simple theoretical model. Future work could examine related issues in larger, empirically-relevant DSGE models. In that regard, the findings of this paper complement the conclusions of AM about the potential implications of incoherency for estimating models with occasionally binding constraints. In particular, AM discuss the potential identification and misspecification issues arising from using estimation methods that neglect incoherent or incomplete regions of the parameter space under RE. Convergence issues due to incoherence may lead researchers to impose overly restrictive prior distributions, further exacerbating these concerns. Estimating models under deviations from RE may alleviate incoherence and incompleteness issues, thus providing an argument for their use in applied work. It is, therefore, worth studying this issue further in empirical applications including the ZLB, such as Aruoba et al. (2018).

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Appendix

A.1 Proof of Proposition 1

Consider Proposition 1 and define $a = \lambda\sigma$, $Q = I_2 - (1 + \beta + \lambda\sigma)K + \beta K^2$ and $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, and let e_j denote the j-th column of the 2 by 2 identity matrix, I_2. We first consider the case $q < 1$.

Case $q < 1$. Because $det(Q + \lambda\sigma\psi I_2) = a(\psi - 1)(a(\psi + 1 - p - q) + (2 - p - q)(1 - \beta(p + q - 1))) > 0$, the PP solution is given by:

$$\hat{\pi}^{PP} = (Q + \lambda\sigma\psi I_2)^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 \end{pmatrix}.$$

The PP solution exists if and only if $\psi\pi_j^{PP} > -\mu$ for $j = 1, 2$. From $\hat{\pi}^{PP}$ we see that π_1^{PP} and π_2^{PP} are linear in ϵ_1 and

$$\frac{\partial \pi_1^{PP}}{\partial \epsilon_1} = \frac{\lambda((1 - q)(1 + a - (p + q - 1)\beta) + a(\psi - 1))}{a(\psi - 1)(a(\psi + 1 - p - q) + (2 - p - q)(1 - \beta(p + q - 1)))} > 0$$

$$\frac{\partial \pi_2^{PP}}{\partial \epsilon_1} = \frac{\lambda(1 - q)(a - \beta(p + q - 1) + 1)}{a(\psi - 1)(a(\psi + 1 - p - q) + (2 - p - q)(1 - \beta(p + q - 1)))} > 0.$$

Thus, PP exists if and only if $\epsilon_1 > \epsilon^{PP} = \max\{\epsilon_1^{PP}, \epsilon_2^{PP}\}$ where ϵ_1^{PP} and ϵ_2^{PP} solve $\psi\pi_1^{PP} = -\mu$ and $\psi\pi_2^{PP} = -\mu$, respectively. We have

$$\epsilon_1^{PP} - \epsilon_2^{PP} = \frac{a(\psi - 1)(a\mu(\psi - 1) + \lambda\epsilon_2\psi)(a(-p - q + \psi + 1) + (-p - q + 2)(1 - \beta(p + q - 1)))}{\lambda(1 - q)\psi(a - \beta(p + q - 1) + 1)(a(\psi - q) + (1 - q)(1 - \beta(p + q - 1)))}$$

and hence $\epsilon_1^{PP} > \epsilon_2^{PP}$. Therefore, the PP solution exists if and only if $\epsilon_1 > \epsilon^{PP} = \epsilon_1^{PP}$ where

$$\epsilon^{PP} = \frac{a^2\mu(\psi - 1)(p + q - \psi - 1)}{\lambda\psi(-(a + 1)q + a\psi + \beta - \beta p + \beta q(p + q - 2) + 1)}$$
$$+ \frac{a(\lambda\epsilon_2(p - 1)\psi + \mu(\psi - 1)(-p - q + 2)(\beta(p + q - 1) - 1)) - \lambda\epsilon_2(p - 1)\psi(\beta(p + q - 1) - 1)}{\lambda\psi(-(a + 1)q + a\psi + \beta - \beta p + \beta q(p + q - 2) + 1)}$$
$$\text{(A1)}$$

and $\lambda\psi(-(a + 1)q + a\psi + \beta - \beta p + \beta q(p + q - 2) + 1) \neq 0$ if $0 < p \leq 1$.

From above, $(Q + \lambda\sigma\psi I_2)^{-1}((\lambda\epsilon_1, \lambda\epsilon_2)')$ is a ZP solution if $\epsilon_1 = \epsilon^{PP}$. If $det(Q + \lambda\sigma\psi e_2 e_2') \neq 0$, then the ZP solution is given by

$$\hat{\pi}^{ZP} = (Q + \lambda\sigma\psi e_2 e_2')^{-1} \begin{pmatrix} \lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 \end{pmatrix}.$$

The ZP solution exists if and only if $\psi\pi_2^{ZP} > -\mu \geq \psi\pi_1^{ZP}$. From $\hat{\pi}^{ZP}$ we see that π_1^{ZP} and

π_2^{ZP} are linear in ϵ_1 and

$$\frac{\partial \pi_1^{ZP}}{\partial \epsilon_1} = \frac{-\lambda((1-q)(a-\beta(p+q-1)+1)+a(\psi-1))}{a(-a(p+q-1)+ap\psi-(\beta(p+q-1)-1)(p(\psi-1)-q-\psi+2))}$$

$$\frac{\partial \pi_2^{ZP}}{\partial \epsilon_1} = \frac{\lambda(q-1)(a-\beta(p+q-1)+1)}{a(-a(p+q-1)+ap\psi-(\beta(p+q-1)-1)(p(\psi-1)-q-\psi+2))}$$

From the last equations it is clear that $\frac{\partial \pi_1^{ZP}}{\partial \epsilon_1} > 0$ and $\frac{\partial \pi_2^{ZP}}{\partial \epsilon_1} > 0$ if and only if $den^{ZP} > 0$ where $den^{ZP} = -(-a(p+q-1)+ap\psi - (\beta(p+q-1)-1)(p(\psi-1)-q-\psi+2)) = a^{-1}det(Q+\lambda\sigma\psi e_2 e_2')$. Solving for ϵ_1^{ZP} and ϵ_2^{ZP} such that $\psi\pi_1^{ZP} = -\mu$ and $\psi\pi_2^{ZP} = -\mu$, respectively, we have

$$\epsilon_1^{ZP} - \epsilon_2^{ZP} = \frac{a(a\mu(\psi-1)+\lambda\epsilon_2\psi)den^{ZP}}{\epsilon_{\Delta ZP,den}}$$

$$\epsilon_{\Delta ZP,den} = (1-q)\lambda\psi(a+\beta(1-p-q)+1)((1-q)(a-\beta(p+q-1)+1)+a(\psi-1)) > 0$$

Therefore, if $den^{ZP} > 0$ then $\epsilon_2^{ZP} < \epsilon_1 \le \epsilon_1^{ZP}$ is necessary and sufficient for existence of ZP; if $den^{ZP} < 0$, $\epsilon_1^{ZP} \le \epsilon_1 < \epsilon_2^{ZP}$ is necessary and sufficient for existence of ZP. Further, we can show:

$$\begin{aligned}
\epsilon_1^{ZP} &= \frac{a^2\mu(\psi-1)(p+q-\psi-1)}{\lambda\psi(-(a+1)q+a\psi+\beta-\beta p+\beta q(p+q-2)+1)} \\
&+ \frac{a(\lambda\epsilon_2(p-1)\psi+\mu(\psi-1)(-p-q+2)(\beta(p+q-1)-1))-\lambda\epsilon_2(p-1)\psi(\beta(p+q-1)-1)}{\lambda\psi(-(a+1)q+a\psi+\beta-\beta p+\beta q(p+q-2)+1)} \\
&= \epsilon^{PP}
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_2^{ZP} &= \frac{a^2\mu(\psi-1)(p+q-1)-\lambda\epsilon_2(p-1)\psi(\beta(p+q-1)-1)}{\lambda(q-1)\psi(-a+\beta(p+q-1)-1)} \\
&+ \frac{a(\lambda\epsilon_2 p\psi+\mu(\psi-1)(2-p-q)(\beta(p+q-1)-1))}{\lambda(q-1)\psi(-a+\beta(p+q-1)-1)}
\end{aligned} \tag{A2}$$

Finally, if $det(Q+\lambda\sigma\psi e_2 e_2') = 0$ ($den^{ZP} = 0$) then $\epsilon^{PP} = \epsilon_2^{ZP}$, and a continuum of ZP solutions exist if $\epsilon_1 = \epsilon^{PP}$ and a ZP solution does not exist if $det(Q+\lambda\sigma\psi e_2 e_2') = 0$ ($den^{ZP} = 0$) and $\epsilon_1 \ne \epsilon^{PP}$.

One can show that the PZ solution does not exist if $det(Q+\lambda\sigma\psi e_1 e_1') = 0$. If $det(Q+\lambda\sigma\psi e_1 e_1') \ne 0$, the PZ solution is given by

$$\hat{\pi}^{PZ} = (Q+\lambda\sigma\psi e_1 e_1')^{-1}\begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2+\lambda\sigma\mu \end{pmatrix}.$$

The PZ solution exists if and only if $\psi\pi_1^{PZ} > -\mu \ge \psi\pi_2^{PZ}$. One can show

$$\frac{\partial \pi_1^{PZ}}{\partial \epsilon_1} = \frac{\lambda(1 - (a+1)q + \beta(q-1)(p+q-1))}{a(a(p-q\psi+q-1) - (\beta(p+q-1)-1)(p-q\psi+q+\psi-2))}$$

$$= \frac{\lambda num_1^{PZ}}{den^{PZ}}$$

$$\frac{\partial \pi_2^{PZ}}{\partial \epsilon_1} = \frac{\lambda(1-q)(a - \beta(p+q-1)+1)}{a(a(p-q\psi+q-1) - (\beta(p+q-1)-1)(p-q\psi+q+\psi-2))}$$

$$= \frac{\lambda num_2^{PZ}}{den^{PZ}}.$$

Clearly $num_2^{PZ} > 0$. Furthermore, it is straightforward to show that $num_1^{PZ} \neq 0$ and $den^{PZ} \neq 0$ is necessary for existence of PZ solution (since $den^{PZ} = det(Q + \lambda\sigma e_1 e_1')$). We now impose $num^{PZ} \neq 0$ and $den^{PZ} \neq 0$ to identify conditions for PZ existence. Now $\frac{\partial den^{PZ}}{\partial \psi} = anum_1^{PZ}$, so den^{PZ} is increasing in ψ if and only if $num_1^{PZ} > 0$. Since den^{PZ} is linear in ψ there exists a unique ψ^{PZ} such that $den^{PZ} = 0$:

$$\psi^{PZ} = 1 + \frac{(1 + a + (1-p-q)\beta)(1-p)}{num_1^{PZ}}$$

Therefore, if $num_1^{PZ} < 0$, $\psi^{PZ} \leq 1$ and den^{PZ} is decreasing in ψ, which implies $den^{PZ} < 0$ for $\psi > 1$. Otherwise, $den^{PZ} < 0$ if $num_1^{PZ} > 0$ and $1 < \psi < \psi^{PZ}$ and $den^{PZ} > 0$ if $num_1^{PZ} > 0$ and $\psi > \psi^{PZ}$.

Solving for ϵ_1^{PZ} and ϵ_2^{PZ} such that $\psi\pi_1^{PZ} = -\mu$ and $\psi\pi_2^{PZ} = -\mu$, respectively, we have

$$\epsilon_1^{PZ} - \epsilon_2^{PZ} = \frac{(a\mu(\psi-1) + \lambda\epsilon_2\psi)den^{PZ}}{\lambda((1-q)(a - \beta(p+q-1)+1))\psi num_1^{PZ}}$$

There are three cases to consider. First, if $den^{PZ} > 0$ (which implies $num_1^{PZ} > 0$), then $\partial\pi_1^{PZ}/\partial\epsilon_1 > 0$, $\partial\pi_2^{PZ}/\partial\epsilon_1 > 0$ and $\epsilon_1^{PZ} > \epsilon_2^{PZ}$ so $\epsilon_1 > \epsilon_1^{PZ} > \epsilon_2^{PZ} \geq \epsilon_1$ is necessary for PZ existence, but not possible. Second, if $den^{PZ} < 0$ and $num_1^{PZ} > 0$, then $\partial\pi_1^{PZ}/\partial\epsilon_1 < 0$, $\partial\pi_2^{PZ}/\partial\epsilon_1 < 0$ and $\epsilon_1^{PZ} < \epsilon_2^{PZ}$, so $\epsilon_1 < \epsilon_1^{PZ} < \epsilon_2^{PZ} \leq \epsilon_1$ is necessary for PZ existence, but not possible. In the third case, $den^{PZ} < 0$ and $num_1^{PZ} < 0$, which implies $\partial\pi_1^{PZ}/\partial\epsilon_1 > 0$, $\partial\pi_2^{PZ}/\partial\epsilon_1 < 0$, and $\epsilon_2^{PZ} < \epsilon_1^{PZ}$ so that $\epsilon_2^{PZ} < \epsilon_1^{PZ} < \epsilon_1$ is necessary and sufficient for PZ existence in this case.

One can show:

$$\epsilon_1^{PZ} - \epsilon^{PP} = \frac{a(p-1)(a\mu(\psi-1) + \lambda\epsilon_2\psi)(a - \beta(p+q-1)+1)}{(num_1^{PZ})\lambda((1-q)(1+a-(-1+p+q)\beta) + a(\psi-1))} \geq 0$$

if PZ exists (since this requires $num_1^{PZ} < 0$). Therefore, if PZ exists then $\epsilon_1 \geq \epsilon^{PP}$ and hence the PP or ZP solution also exists.

From above, $(Q + \lambda\sigma\psi e_2 e_2')^{-1}((\lambda\epsilon_1 + \lambda\sigma\mu, \lambda\epsilon_2)')$ is a ZZ solution if $\epsilon_1 = \epsilon_2^{ZP}$ and $det(Q +$

$\lambda\sigma\psi e_2 e_2') \neq 0$. If $det(Q) \neq 0$, the ZZ solution is given by

$$\hat{\pi}^{ZZ} = (Q)^{-1} \begin{pmatrix} \lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The ZZ solution exists if and only if $\psi\pi_j^{ZZ} \leq -\mu$ for $j = 1, 2$. One can show

$$\frac{\partial\pi_1^{ZZ}}{\partial\epsilon_1} = \frac{\lambda(1 - (a+1)q + \beta(q-1)(p+q-1))}{a(a(p+q-1) - (p+q-2)(\beta(p+q-1)-1))}$$
$$= \frac{\lambda num_1^{ZZ}}{aden^{ZZ}}$$
$$\frac{\partial\pi_2^{ZZ}}{\partial\epsilon_1} = \frac{\lambda(1-q)(a - \beta(p+q-1)+1)}{a(a(p+q-1) - (p+q-2)(\beta(p+q-1)-1))}$$
$$= \frac{\lambda num_2^{ZZ}}{aden^{ZZ}}.$$

where $det(Q) = aden^{ZZ}$. Clearly, $num_2^{ZZ} > 0$. We can further show that $-num_1^{ZZ} = den^{ZZ} + (1-p)(1+a - (p+q-1)\beta) \geq den^{ZZ}$. Hence $den^{ZZ} > 0$ implies $num_1^{ZZ} < 0$. Solving for ϵ_1^{ZZ} and ϵ_2^{ZZ} such that $\psi\pi_1^{ZZ} = -\mu$ and $\psi\pi_2^{ZZ} = -\mu$, respectively, we have

$$\epsilon_1^{ZZ} - \epsilon_2^{ZZ} = \frac{aden^{ZZ}(a\mu(\psi-1) + \lambda\epsilon_2\psi)}{\lambda num_1^{ZZ}\psi(1-q)(a - \beta(p+q-1)+1)}$$

if $num^{ZZ} \neq 0$. There are the following cases to consider. First, if $den^{ZZ} > 0$ (which implies $num_1^{ZZ} < 0$) then $\partial\pi_1^{ZZ}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ}/\partial\epsilon_1 > 0$, $\epsilon_2^{ZZ} > \epsilon_1^{ZZ}$, so that ZZ existence requires $\epsilon_2^{ZZ} \geq \epsilon_1 \geq \epsilon_1^{ZZ}$. Second, if $den^{ZZ} < 0$ and $num_1^{ZZ} > 0$ then $\partial\pi_1^{ZZ}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ}/\partial\epsilon_1 < 0$, $\epsilon_2^{ZZ} > \epsilon_1^{ZZ}$, so that ZZ existence requires $\epsilon_1 \geq \epsilon_2^{ZZ} > \epsilon_1^{ZZ}$. In the third case, $den^{ZZ} < 0$ and $num_1^{ZZ} < 0$ which implies $\partial\pi_1^{ZZ}/\partial\epsilon_1 > 0$, $\partial\pi_2^{ZZ}/\partial\epsilon_1 < 0$, $\epsilon_1^{ZZ} > \epsilon_2^{ZZ}$, so that ZZ existence requires $\epsilon_1^{ZZ} \geq \epsilon_1 \geq \epsilon_2^{ZZ}$. If $num_1^{ZZ} = 0$ and $det(Q) \neq 0$ then a ZZ exists if and only if $\epsilon_1 \geq \epsilon_2^{ZZ}$. Finally, if $det(Q) = 0$ ($den^{ZZ} = 0$) and $\epsilon_1 = \epsilon_2^{ZP}$ then a continuum of ZZ solutions exist, and if $det(Q) = 0$ ($den^{ZZ} = 0$) and $\epsilon_1 \neq \epsilon_2^{ZP}$ then a ZZ solution does not exist.

Now it can be shown that $\epsilon_2^{ZZ} = \epsilon_2^{ZP}$ and

$$\epsilon_1^{ZZ} - \epsilon^{PP} = \frac{a(p-1)(a\mu(\psi-1) + \lambda\epsilon_2\psi)(a - \beta(p+q-1)+1)}{\lambda num_1^{ZZ}((1-q)(1+a - (-1+p+q)\beta) + a(\psi-1))} \geq 0$$

if $num_1^{ZZ} < 0$. Since $\epsilon_2^{ZZ} = \epsilon_2^{ZP}$ and existence of ZZ only hinges on $\epsilon_1 \geq \epsilon_1^{ZZ}$ if $num_1^{ZZ} < 0$ and therefore $\epsilon_1^{ZZ} \geq \epsilon^{PP}$, it follows that a PP or ZP solution exists if a ZZ solution exists and $\epsilon_1 > \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$.

From the analysis above, a REE exists only if $\epsilon_1 \geq \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$. Further, if $\epsilon_1 \geq \epsilon^{PP}$ then a PP or ZP exists because $det(Q + \lambda\sigma\psi I_2) > 0$. If $\epsilon^{PP} > \epsilon_2^{ZP}$, then $det(Q + a\psi e_2 e_2') = aden^{ZP} \neq 0$ and therefore a PP, ZP or ZZ solution exists if, in addition, $\epsilon_1 \geq \epsilon_2^{ZP}$. We conclude that an REE exists if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{REE} = \min\{\epsilon^{PP}, \epsilon_2^{ZP}\} \tag{A3}$$

42

where ϵ^{PP} and ϵ_2^{ZP} are defined in (A1) and (A2), respectively.

Case $q = 1$. Here we show that Proposition 1 nests Proposition 5 of AM as a special case. Specifically, we compute the condition from $\lim_{q\to1} \bar{\epsilon}_{REE}$ and show that this recovers the result in Proposition 5 of AM. Alternatively, we could repeat the preceding analysis in the model with $q = 1$, but this gives the same result.[28]

Define $\theta = \frac{(1-p)(1-p\beta)}{\lambda\sigma p} = \frac{(1-p)(1-p\beta)}{ap}$. From the preceding analysis, an REE exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{REE} = \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$ where ϵ_2^{ZP} can be expressed as $\epsilon_2^{ZP} = \chi(1-q)^{-1}$. In the limit $q \to 1$ we have:

$$
\epsilon^{PP} = \mu\left(\frac{a(p-\psi)}{\lambda\psi} - \frac{pa\theta}{\lambda\psi}\right) + \frac{\lambda\epsilon_2(p-1)(a-\beta p+1)}{a\lambda(\psi-1)}
$$

$$
\chi = \frac{(p(1+a+\beta) - p^2\beta - 1)(a\mu(\psi-1) + \lambda\epsilon_2\psi)}{(1+a-p\beta)\psi\lambda}.
$$

Now, $p(1 + a + \beta) - 1 - p^2\beta < 0$ if and only if $\theta > 1$. Therefore, $\bar{\epsilon}_{REE} = \epsilon_2^{ZP} \to -\infty$ as $q \to 1$ if $\theta > 1$. We conclude that any value of ϵ_1 ensures existence of a solution when $\theta > 1$ and $q = 1$. If $\theta < 1$, then $\chi \to +\infty$ and $\bar{\epsilon}_{REE} = \epsilon^{PP}$, and $\epsilon_2^{ZP} \geq \epsilon^{PP} = \bar{\epsilon}_{REE}$ if $\theta = 1$.[29]

Now we show that our conditions recover Proposition 5 in AM. First, we have $\mu = log(r\pi_*) > 0$ which implies $r^{-1} \leq \pi_*$ where r and π_* are the steady state gross real interest rate and inflation rate, respectively. Further, we set $\epsilon_2 = 0$ and $\epsilon_1 = -\sigma\hat{M}_{t+1|t} = \sigma pr_L$. The critical threshold, ϵ^{PP} becomes

$$
-r_L \leq \mu\left(\frac{\theta}{\psi} + \frac{(\psi-p)}{p\psi}\right)
$$

Thus, a solution exists if and only if either $\theta > 1$ or $\theta \leq 1$ and $-r_L \leq \mu\left(\frac{\theta}{\psi} + \frac{(\psi-p)}{p\psi}\right)$ as in AM.

A.2 Proof of Proposition 2

Consider (1)-(3) and suppose that expectations evolve according to

$$
\hat{E}_t(y_{t+1}) = \hat{E}_{t-1}(y_t) + g_{y,t}\left(y_{t-k} - \hat{E}_{t-1}(y_t)\right) \tag{A4}
$$

with $k = 0, 1$, $g_{y,t} \in [0, 1]$, given some initial condition $\hat{E}_0(y_1) = a_{y0}$. To demonstrate coherence and completeness, we begin by showing that there is a unique solution for π_0, x_0, i_0 given initial expectations and exogenous shocks. Substituting (1), (3), $E_0 y_1 = a_{y0}$ into (2) we have

$$
\pi_0 = -\lambda\sigma\max\{\psi\pi_t, -\mu\} + \lambda(a_{x0} + \sigma a_{\pi0} + \epsilon_0) + \beta a_{\pi0} + u_0 \tag{A5}
$$

---

[28]Mathematica routine available on request.

[29]The $\theta = 1$ case arises if $a = \frac{(1-p)(1+\beta(1-p-q))}{p}$ and $q = 1$. To compute ϵ_2^{ZP}, set $a = \frac{(1-p)(1+\beta(1-p-q))}{p}$ and compute $\lim_{q\to1} \epsilon_2^{ZP}$.

Given $a_{x0}, a_{\pi0}, u_0, \epsilon_0$ there is a unique solution for π_0 obtained from (A5) if $\psi > 0$. The unique solution for i_0, x_0 is obtained from the Phillips curve (2) and interest rate rule (3). Therefore, for $t \geq 0$, we have:

$$\hat{E}_{t-1}\pi_t = \beta^{-1}(\pi_{t-1} - \lambda x_{t-1} - u_{t-1}) \tag{A6}$$

$$\hat{E}_{t-1}x_t = (1 + \sigma\lambda\beta^{-1})x_{t-1} - \sigma\beta^{-1}\pi_{t-1} + \sigma i_{t-1} - \epsilon_{t-1} + \sigma\beta^{-1}u_{t-1} \tag{A7}$$

Substituting (A6)-(A7) into (A4), and then (A4) into (1)-(3), the model can be written in the form

$$A_{11}Y_{1t} + A_{12}Y_{2t} + A_{12}^*Y_{2t}^* = B_{10}X_{0t} + B_{11}Y_{t-1} + B_{11}^*Y_{2t-1}^* + \epsilon_{1t} \tag{A8}$$

$$A_{21}Y_{1t} + A_{22}Y_{2t} + A_{22}^*Y_{2t}^* = B_{20}X_{0t} + B_{21}Y_{t-1} + B_{21}^*Y_{2t-1}^* + \epsilon_{2t} \tag{A9}$$

$$Y_{2t} = \max\{Y_{2t}^*, -\mu\} \tag{A10}$$

for $t > 0$, where $Y_{1t} = (x_t, \pi_t)'$, $Y_{2t}^* = i_t^* = \psi\pi_t$, $Y_t = (Y_{1t}', Y_{2t})'$, and X_{0t} are exogenous shocks.

Case $k = 0$. Under contemporaneous information, we have

$$A_{11} = \begin{pmatrix} 1 - g_x & -\sigma g_x \\ -\lambda & 1 - \beta g_\pi \end{pmatrix} \qquad A_{12} = \begin{pmatrix} \sigma & 0 \end{pmatrix}' \qquad A_{12}^* = \begin{pmatrix} 0 & 0 \end{pmatrix}'$$

$$A_{21} = \begin{pmatrix} 0 & -\psi \end{pmatrix} \qquad\qquad A_{22} = 0 \qquad\qquad\qquad A_{22}^* = 1$$

Following Proposition 1 of Mavroeidis (2021), we have coherence and completeness if and only if

$$1 + \frac{\sigma\lambda\psi}{(1 - g_{x,t})(1 - \beta g_{\pi,t}) - \lambda\sigma g_{\pi,t}} > 0$$

It is easy to see that we only have coherence and completeness under the Taylor principle ($\psi > 1$) if $g_{x,t}$, $g_{\pi,t}$, or λ are not too large. For instance, as $\lambda \to \infty$ we have the coherence and completeness condition: $g_{\pi,t} > \psi$, which only holds if we violate the Taylor Principle. Further, if agents have "naive" expectations (i.e. if $g_{x,t} = g_{\pi,t} = 1$) then we have the coherence condition: $\psi < 1$. We need to restrict gain parameters and the slope of the Phillips curve to have coherence under the Taylor principle, adaptive expectations and contemporanous information ($k = 0$).

Case $k = 1$. Under lagged information, we have

$$A_{11} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \qquad\qquad A_{12} = \begin{pmatrix} \sigma & 0 \end{pmatrix}' \qquad A_{12}^* = \begin{pmatrix} 0 & 0 \end{pmatrix}'$$

$$A_{21} = \begin{pmatrix} 0 & -\psi \end{pmatrix} \qquad\qquad A_{22} = 0 \qquad\qquad\qquad A_{22}^* = 1$$

Following Proposition 1 of Mavroeidis (2021), we have coherence and completeness if and only if

$$1 + \sigma\lambda\psi > 0$$

which holds for all $\psi > 0$.

A.3 Proof of Proposition 3

The proof of Proposition 3 is a straightforward extension of the proof of Proposition 1. Define $\bar{q} = Pr(s_t = 2) = (1-p)/(2-p-q)$. The regime-specific levels of inflation in RPE i, $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, are given by fixed point restrictions that have the same basic form as the REE fixed point restrictions except we replace q with $\bar{q}$ and p with $1 - \bar{q}$. Therefore, RPE will exist if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{RPE} = \min\{\epsilon^{PP,RPE}, \epsilon_2^{ZP,RPE}\}, \tag{A11}$$

where $\epsilon^{PP,RPE}, \epsilon_2^{ZP,RPE}$ have the same form as $\epsilon^{PP}, \epsilon_2^{ZP}$ given in (A1),(A2) except we replace q and p with $\bar{q}$ and $1 - \bar{q}$, respectively. In the special case $q = 1$ (which implies $\bar{q} = 1$), we have $\bar{\epsilon}_{RPE} = -\infty$, as the PP solution exists if and only if $\epsilon_1 > -\mu(1 + \lambda\sigma\psi)(\lambda\psi)^{-1} + (1 + \lambda\sigma)(\lambda\sigma(1-\psi))^{-1}\epsilon_2 = \epsilon^{PP,RPE}$ and the ZP exists if and only if $\epsilon_1 \leq \epsilon^{PP,RPE}$.

For $q < 1$, one can show:

$$\begin{aligned}
\epsilon^{PP} - \epsilon^{PP,RPE} &= \Xi_{PP}(1 - p - q) \\
\epsilon^{ZP} - \epsilon^{ZP,RPE} &= \Xi_{ZP}(1 - p - q)
\end{aligned}$$

$\Xi_{PP} = \frac{a(1+a-\beta(p+q-2))(1-p)(\psi-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{\lambda\psi(a(1-\psi)(2-p-q)+(a+1)(q-1))((1-q)(1+a-\beta(p+q-1))+a(\psi-1))} \leq 0$
and $\Xi_{ZP} = \frac{a(a\mu(\psi-1)+\lambda\epsilon_2\psi)(1+a-\beta(p+q-2))}{\lambda(a+1)(q-1)\psi(1+a-\beta(p+q-1))} < 0$. Hence, $\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if and only if $p + q \geq 1$.

A.4 Proof of Proposition 4

Consider Proposition 4 and define $a = \lambda\sigma$, $\delta = (M-1)(1 - M_f\beta) + \lambda\sigma N$, $Q = I_2 - (M + M_f\beta + \lambda\sigma N)K + \beta M M_f K^2$ and $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, and let e_j denote the j-th column of the 2 by 2 identity matrix, I_2. We consider the case $q < 1$ first.

Case $q < 1$. Since $det\,(Q + \lambda\sigma\psi I_2) = ((1 - M(p+q-1))(1 - M_f\beta(p+q-1)) + a(\psi + N(1-p-q)))((1-M)(1-M_f\beta) + a(\psi - N)) > 0$, the PP solution is given by:

$$\hat{\pi}^{PP,BR} = (Q + \lambda\sigma\psi I_2)^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 \end{pmatrix}.$$

The PP solution exists if and only if $\psi\pi_j^{PP,BR} > -\mu$ for $j = 1, 2$. From $\hat{\pi}^{PP,BR}$ we see that $\pi_1^{PP,BR}$ and $\pi_2^{PP,BR}$ are linear in ϵ_1 and

$$\frac{\partial \pi_1^{PP,BR}}{\partial \epsilon_1} = \frac{num_1^{PP,BR}}{den^{PP,BR}} > 0$$

$$\frac{\partial \pi_2^{PP,BR}}{\partial \epsilon_1} = \frac{num_2^{PP,BR}}{den^{PP,BR}} > 0.$$

where

$$num_1^{PP,BR} = \lambda(a\psi + \beta MM_f(p(q-1) + q(q-1) + 1) - Mq - q(\beta M_f + aN) + 1) > 0$$
$$num_2^{PP,BR} = \lambda(q-1)(\beta M_f(M(p+q) - 1) - M - aN) > 0$$
$$den^{PP,BR} = (a(\psi - N) + (1-M)(1 - \beta M_f))den_1^{PP,BR} > 0$$
$$den_1^{PP,BR} = a\psi + M(p+q-1)(\beta M_f(p+q-1) - 1)$$
$$+ \beta M_f - (p+q)(\beta M_f + aN) + aN + 1 > 0$$

Thus, PP exists if and only if $\epsilon_1 > \epsilon^{PP,BR} = \max\{\epsilon_1^{PP,BR}, \epsilon_2^{PP,BR}\}$ where $\epsilon_1^{PP,BR}$ and $\epsilon_2^{PP,BR}$ solve $\psi\pi_1^{PP,BR} = -\mu$ and $\psi\pi_2^{PP,BR} = -\mu$, respectively. We have

$$\epsilon_1^{PP,BR} - \epsilon_2^{PP,BR} =$$
$$\frac{den^{PP,BR}(\mu(a(\psi - N) + (1-M)(1 - \beta M_f)) + \lambda\epsilon_2\psi)}{\lambda\psi num_1^{PP,BR} num_2^{PP,BR}} \geq 0$$

Therefore, the PP solution exists if and only if $\epsilon_1 > \epsilon^{PP,BR} = \epsilon_1^{PP,BR}$ where

$$\epsilon^{PP,BR} = \frac{\eta_1 \eta_2 \eta_3}{\psi num_1^{PP,BR}} \tag{A12}$$
$$\eta_1 = a(\psi - N) + (1-M)(1 - M_f\beta) > 0$$
$$\eta_2 = (-(p+q)(aN + \beta M_f) + a(N + \psi) + M(p+q-1)(\beta M_f(p+q-1) - 1) + \beta M_f + 1)$$
$$\eta_3 = \frac{\lambda\epsilon_2(1-p)\psi(-aN + \beta M_f(M(p+q) - 1) - M)}{den^{PP,BR}} - \mu$$

From above, $(Q + \lambda\sigma\psi I_2)^{-1}((\lambda\epsilon_1, \lambda\epsilon_2)')$ is a ZP solution if $\epsilon_1 = \epsilon^{PP,BR}$. If $det(Q + \lambda\sigma\psi e_2 e_2') \neq 0$, then the ZP solution is given by

$$\hat{\pi}^{ZP,BR} = (Q + \lambda\sigma\psi e_2 e_2')^{-1} \begin{pmatrix} \lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 \end{pmatrix}.$$

The ZP solution exists if and only if $\psi\pi_2^{ZP,BR} > -\mu \geq \psi\pi_1^{ZP,BR}$. From $\hat{\pi}^{ZP,BR}$ we see that

$\pi_1^{ZP,BR}$ and $\pi_2^{ZP,BR}$ are linear in ϵ_1 and

$$\frac{\partial \pi_1^{ZP,BR}}{\partial \epsilon_1} = \frac{\lambda(a(\psi - Nq) + \beta M_f(M(p(q-1) + q(q-1) + 1) - q) - Mq + 1)}{den^{ZP,BR}}$$

$$\frac{\partial \pi_2^{ZP,BR}}{\partial \epsilon_1} = \frac{\lambda(1-q)(aN + \beta M_f(1 - M(p+q)) + M)}{den^{ZP,BR}}$$

From the last equations it is clear that $\frac{\partial \pi_1^{ZP}}{\partial \epsilon_1} > 0$ and $\frac{\partial \pi_2^{ZP}}{\partial \epsilon_1} > 0$ if and only if $den^{ZP,BR} > 0$ where $den^{ZP,BR} = -M(aN(p+q-1)(\beta M_f(p+q)-2) + a\psi(\beta M_f(-p(p+q-1)+q-1)+p) + (\beta M_f-1)(p+q)(\beta M_f(p+q-1)-1)) - a\psi(aNp + \beta M_f p - 1) + (aN + \beta M_f - 1)(aN(p+q-1) + \beta M_f(p+q-1)-1) - M^2(\beta M_f - 1)(-(p+q-1))(\beta M_f(p+q-1)-1) = det(Q + \lambda\sigma e_2 e_2')$. Solving for $\epsilon_1^{ZP,BR}$ and $\epsilon_2^{ZP,BR}$ such that $\psi\pi_1^{ZP,BR} = -\mu$ and $\psi\pi_2^{ZP,BR} = -\mu$, respectively, we have

$$\epsilon_1^{ZP,BR} - \epsilon_2^{ZP,BR} = \frac{(\mu((1-M)(1-M_f\beta) + a(\psi - N)) + \lambda\epsilon_2\psi)den^{ZP,BR}}{\lambda\epsilon_{\Delta ZP,BR}}$$

$$\epsilon_{\Delta ZP,BR} = (1-q)num_1^{ZP,BR}(M + aN + M_f\beta(1 - M(p+q))) > 0$$

$$num_1^{ZP,BR} = \psi(a(\psi - Nq) + \beta M_f(M(p(q-1) + q(q-1) + 1) - q) - Mq + 1) > 0$$

Therefore, if $den^{ZP,BR} > 0$ then $\epsilon_2^{ZP,BR} < \epsilon_1 \le \epsilon_1^{ZP,BR}$ is necessary and sufficient for existence of ZP; if $den^{ZP,BR} < 0$, $\epsilon_1^{ZP,BR} \le \epsilon_1 < \epsilon_2^{ZP,BR}$ is necessary and sufficient for existence of ZP. Further, we can show:

$$\epsilon_1^{ZP,BR} = \epsilon^{PP,BR}$$

and

$$\begin{aligned}
\epsilon_2^{ZP,BR} = &\frac{\mu\eta_1(-(p+q)(aN + \beta M_f) + aN + M(p+q-1)(\beta M_f(p+q-1) - 1) + \beta M_f + 1)}{\lambda(q-1)\psi(aN - \beta M M_f(p+q) + M + \beta M_f)} \\
&- \frac{\psi\epsilon_2\lambda(aNp + \beta M_f(M(-p(p+q-1) + q - 1) + p) + Mp - 1)}{\lambda(q-1)(aN - \beta M M_f(p+q) + M + \beta M_f)}
\end{aligned} \tag{A13}$$

Finally, if $det(Q + \lambda\sigma\psi e_2 e_2') = 0$ ($den^{ZP,BR} = 0$) then $\epsilon^{PP,BR} = \epsilon_2^{ZP,BR}$, and a continuum of ZP solutions exist if $\epsilon_1 = \epsilon^{PP,BR} = \epsilon_2^{ZP,BR}$ and no ZP solution exists if $det(Q + \lambda\sigma\psi e_2 e_2') = 0$ ($den^{ZP,BR} = 0$) and $\epsilon_1 \ne \epsilon^{PP,BR}$.

It is straightforward to show that the PZ solution does not exist if $det(Q + \lambda\sigma\psi e_1 e_1') = 0$. If $det(Q + \lambda\sigma\psi e_1 e_1') \ne 0$, the PZ solution is given by

$$\hat{\pi}^{PZ,BR} = (Q + \lambda\sigma\psi e_1 e_1')^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The PZ solution exists if and only if $\psi\pi_1^{PZ,BR} > -\mu \ge \psi\pi_2^{PZ,BR}$. One can show

$$\frac{\partial \pi_1^{PZ,BR}}{\partial \epsilon_1} = \frac{\lambda(1 - (M + aN)q + M_f(M + Mp(-1 + q) - q + M(-1 + q)q)\beta))}{den^{PZ,BR}}$$

$$= \frac{\lambda num_1^{PZ,BR}}{den^{PZ,BR}}$$

$$\frac{\partial \pi_2^{PZ,BR}}{\partial \epsilon_1} = \frac{\lambda(1 - q)(M + aN + M_f(1 - M(p + q))\beta)}{den^{PZ,BR}}$$

$$= \frac{\lambda num_2^{PZ,BR}}{den^{PZ,BR}}.$$

where $den^{PZ,BR} = det(Q + \lambda \sigma \psi e_1 e_1') = -M(aN(p+q-1)(\beta M_f(p+q) - 2) + a\psi(\beta M_f(p-1) - \beta M_f q(p+q-1) + q) + (\beta M_f - 1)(p+q)(\beta M_f(p+q-1) - 1)) + (aN + \beta M_f - 1)(aN(p+q-1) + \beta M_f(p+q-1) - 1) - a\psi(aNq + \beta M_f q - 1) + M^2(\beta M_f - 1)(p+q-1)(\beta M_f(p+q-1) - 1)$. Clearly $num_2^{PZ,BR} > 0$. Furthermore, it is straightforward to show that $num_1^{PZ,BR} \neq 0$ is necessary for existence of PZ solution. Solving for $\epsilon_1^{PZ,BR}$ and $\epsilon_2^{PZ,BR}$ such that $\psi \pi_1^{PZ,BR} = -\mu$ and $\psi \pi_2^{PZ,BR} = -\mu$, respectively, we have

$$\epsilon_1^{PZ,BR} - \epsilon_2^{PZ,BR} = \frac{(\eta_1 \mu + \psi \lambda \epsilon_2)den^{PZ,BR}}{\lambda(1 - q)\psi(M + aN + M_f \beta(1 - M(p + q)))num_1^{PZ,BR}}$$

if $num^{PZ,BR} \neq 0$. There are four cases to consider. First, if $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} > 0$, then $\partial \pi_1^{PZ,BR}/\partial \epsilon_1 > 0$, $\partial \pi_2^{PZ,BR}/\partial \epsilon_1 > 0$ and $\epsilon_1^{PZ,BR} > \epsilon_2^{PZ,BR}$ so $\epsilon_1 > \epsilon_1^{PZ,BR} > \epsilon_2^{PZ,BR} \geq \epsilon_1$ is necessary for PZ existence, but not possible. Second, if $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} > 0$, then $\partial \pi_1^{PZ,BR}/\partial \epsilon_1 < 0$, $\partial \pi_2^{PZ,BR}/\partial \epsilon_1 < 0$ and $\epsilon_1^{PZ,BR} < \epsilon_2^{PZ,BR}$, so $\epsilon_1 < \epsilon_1^{PZ,BR} < \epsilon_2^{PZ,BR} \leq \epsilon_1$ is necessary for PZ existence, but not possible. In the third case, $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} < 0$, which implies $\partial \pi_1^{PZ,BR}/\partial \epsilon_1 > 0$, $\partial \pi_2^{PZ,BR}/\partial \epsilon_1 < 0$, and $\epsilon_2^{PZ,BR} < \epsilon_1^{PZ,BR}$ so that $\epsilon_2^{PZ,BR} < \epsilon_1^{PZ,BR} < \epsilon_1$ is necessary and sufficient for PZ existence in this case. In the fourth case, $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$. One can show:

$$den^{PZ,BR} = \delta(p - 1)(M + aN + M_f \beta(1 - M(p + q)))$$
$$+ num_1^{PZ,BR} \eta_1$$

Hence, if $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$ then $\delta < 0$, since $M + aN + M_f \beta(1 - M(p+q)) > 0$. Now, for PZ to exist it must be the case that

$$\pi_1^{PZ,BR} - \pi_2^{PZ,BR} = \frac{-\psi \delta \lambda \epsilon_1 - \psi \eta_1(\lambda \epsilon_2 + a\mu)}{den^{PZ,BR}} > 0$$

$$\rightarrow \quad \epsilon_1 > \frac{\eta_1(\lambda \epsilon_2 + a\mu)}{-\delta \lambda} > 0$$

where the last three inequalities hold if $\delta < 0$ and $den^{PZ,BR} > 0$. Also, if $den^{PZ,BR} > 0$ then

$\epsilon_1 \le \epsilon_2^{PZ,BR}$ is necessary for existence of PZ solution. If $\delta < 0$, we have:

$$\epsilon_2^{PZ,BR} = \frac{\lambda\epsilon_2\psi(aNp - a\psi + M(\beta M_f(-p(p+q-1)+q-1)+p) + \beta M_f p - 1)}{\lambda(1-q)\psi(aN + \beta M_f(1 - M(p+q)) + M)}$$
$$- \frac{\mu\eta_1(-(p+q)(aN+\beta M_f) + a(N+\psi) + M(p+q-1)(\beta M_f(p+q-1)-1) + \beta M_f + 1)}{\lambda(1-q)\psi(aN + \beta M_f(1-M(p+q)) + M)} < 0$$

Since $\epsilon_1 > 0 > \epsilon_2^{PZ,BR} \ge \epsilon_1$ does not hold, the PZ solution does not exist if $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$.

Hence, a PZ solution can only exist if $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} < 0$ and $\epsilon_1 > \epsilon_1^{PZ,BR}$. One can show:

$$\epsilon_1^{PZ,BR} - \epsilon^{PP,BR} = \frac{\psi a(1-p)(aN + M + M_f\beta(1-M(p+q)))(\lambda\psi\epsilon_2 + \mu\eta_1)}{-\lambda num_1^{ZP,BR} num_1^{PZ,BR}} \ge 0$$

if PZ exists (since this requires $num_1^{PZ,BR} < 0$). Therefore, if the PZ exists then $\epsilon_1 \ge \epsilon^{PP,BR}$ and hence the PP or ZP solution also exists.

From above, $(Q + \lambda\sigma\psi e_2 e_2')^{-1}((\lambda\epsilon_1 + \lambda\sigma\mu, \lambda\epsilon_2)')$ is a ZZ solution if $\epsilon_1 = \epsilon_2^{ZP,BR}$ and $det(Q + \lambda\sigma\psi e_2 e_2') \ne 0$ ($den^{ZP,BR} \ne 0$). If $det(Q) \ne 0$ then the ZZ solution is given by

$$\hat{\pi}^{ZZ,BR} = (Q)^{-1}\begin{pmatrix} \lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The ZZ solution exists if and only if $\psi\pi_j^{ZZ,BR} \le -\mu$ for $j = 1, 2$. One can show

$$\frac{\partial\pi_1^{ZZ,BR}}{\partial\epsilon_1} = \frac{\lambda((1-(M+aN)q + M_f(M + Mp(-1+q) - q + M(-1+q)q)\beta))}{den^{ZZ,BR}}$$
$$= \frac{\lambda num_1^{ZZ,BR}}{den^{ZZ,BR}}$$
$$\frac{\partial\pi_2^{ZZ,BR}}{\partial\epsilon_1} = \frac{\lambda(1-q)(M + aN + M_f\beta(1-M(p+q)))}{den^{ZZ,BR}}$$
$$= \frac{\lambda num_2^{ZZ,BR}}{den^{ZZ,BR}}.$$

where $den^{ZZ,BR} = -\delta(1 + aN + M_f\beta - (p+q)(aN + M_f\beta) + M(p+q-1)(M_f\beta(p+q-1)-1)) = det(Q)$ and clearly $num_2^{ZZ,BR} > 0$. Solving for $\epsilon_1^{ZZ,BR}$ and $\epsilon_2^{ZZ,BR}$ such that $\psi\pi_1^{ZZ,BR} = -\mu$ and $\psi\pi_2^{ZZ,BR} = -\mu$, respectively, we have

$$\epsilon_1^{ZZ,BR} - \epsilon_2^{ZZ,BR} = \frac{den^{ZZ,BR}(\eta_1\mu + \lambda\psi\epsilon_2)}{\lambda(1-q)\psi(M + aN + M_f\beta(1 - M(p+q)))num_1^{ZZ,BR}}$$

if $num_1^{ZZ,BR} \ne 0$. There are the following cases to consider. First, if $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} < 0$ then $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 > 0$, $\epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$, so that ZZ existence requires $\epsilon_2^{ZZ,BR} \ge \epsilon_1 \ge \epsilon_1^{ZZ,BR}$. Second, if $den^{ZZ,BR} < 0$ and $num_1^{ZZ,BR} > 0$ then $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 < 0$, $\epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$, so that ZZ existence requires

$\epsilon_1 \geq \epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$. In the third case, $den^{ZZ,BR} < 0$ and $num_1^{ZZ,BR} < 0$ which implies $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 > 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 < 0$, $\epsilon_1^{ZZ,BR} > \epsilon_2^{ZZ,BR}$, so that ZZ existence requires $\epsilon_1^{ZZ,BR} \geq \epsilon_1 \geq \epsilon_2^{ZZ,BR}$.

Now it can be shown that $\epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$ and

$$\epsilon_1^{ZZ,BR} - \epsilon^{PP,BR} = \frac{-a(1-p)(M + aN + M_f\beta(1 - M(p+q)))(\psi\lambda\epsilon_2 + \eta_1\mu)}{\lambda num_1^{ZZ,BR}\eta_4} \geq 0$$

if $num_1^{ZZ,BR} < 0$, where

$$\eta_4 = (1-q)(aN + \beta M_f(1 - M(p+q)) + M) + \\ a(\psi - N) + (1-M)(1 - \beta M_f) > 0$$

Since $\epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$ and existence of ZZ in the first three cases only hinges on $\epsilon_1 \geq \epsilon_1^{ZZ,BR}$ if $num_1^{ZZ,BR} < 0$ it follows that the ZP or PP solution will exist if the ZZ solution exists in the first three cases if $\epsilon_1 > \max\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\}$.

In the fourth case, $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} > 0$. One can show that:

$$den^{ZZ,BR} = -\delta(-\delta + (2 - p - q)(M + aN + M_f\beta(1 - (p+q)M)))$$
$$num_1^{ZZ,BR} = -\delta^{-1}den^{ZZ,BR} + \eta_5$$
$$= -\delta + (1-q)(M + aN + M_f\beta(1 - (p+q)M))$$
$$\eta_5 = (p-1)(M + aN + M_f\beta(1 - (p+q)M)) \leq 0$$

Therefore, $\delta < 0$ if and only if the fourth case ($num_1^{ZZ,BR} > 0$ and $den^{ZZ,BR} > 0$) applies. In the fourth case, $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 > 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 > 0$, $\epsilon_1^{ZZ,BR} > \epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$, so that ZZ existence requires $\epsilon_2^{ZP,BR} \geq \epsilon_1$. It is furthermore straightforward to show that if $num_1^{ZZ,BR} = 0$ and $det(Q) \neq 0$ then a ZZ exists if and only if $\epsilon_1 \geq \epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$. Finally, if $det(Q) = 0$ ($den^{ZZ,BR} = 0$) and $\epsilon_1 = \epsilon_2^{ZP,BR}$ then a continuum of ZZ solutions exist and if $det(Q) = 0$ ($den^{ZZ,BR} = 0$) and $\epsilon_1 \neq \epsilon_2^{ZP,BR}$ then a ZZ solution does not exist.

From the analysis above, if a BRE exists then $\delta \geq 0$ and $\epsilon_1 \geq \min\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\}$ or $\delta < 0$. Further, if $\epsilon_1 \geq \epsilon^{PP,BR}$ then a PP or ZP exists because $det(Q + \lambda\sigma\psi I_2) > 0$. If $\epsilon^{PP,BR} > \epsilon_2^{ZP,BR}$, then $den^{ZP,BR} = det(Q + \lambda\sigma e_2 e_2') \neq 0$ and therefore a PP, ZP or ZZ solution exists if, in addition, $\epsilon_1 \geq \epsilon_2^{ZP,BR}$. If $\delta < 0$, then a ZZ exists for $\epsilon_1 \leq \epsilon_2^{ZP,BR}$. We conclude that a BRE exists if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{BR} := \begin{cases} \min\left\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\right\}, & \text{if } \delta \geq 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \tag{A14}$$

where $\epsilon^{PP,BR}$ and $\epsilon_2^{ZP,BR}$ are defined in (A12) and (A13), respectively.

Case $q = 1$. We begin by noting that $\epsilon_2^{ZP,BR}$ from (A13) can be expressed as

$$\epsilon_2^{ZP,BR} = (q-1)^{-1}\chi_{BR}$$

50

where, if $q = 1$, and $\chi^1 = -\delta + (1-p)(aN + M(1 - M_f\beta p) + M_f\beta(1 - M)) \neq 0$:

$$\chi_{BR} = \frac{\chi^1(\psi\lambda\epsilon_2 + \mu((1-M)(1 - M_f\beta) + a(\psi - N)))}{\lambda\psi(aN + M_f\beta(1-M) + M(1 - M_f\beta))}$$

where $\psi(aN + M_f\beta(1-M) + M(1 - M_f\beta)) > 0$, $\psi\lambda\epsilon_2 + \mu((1-M)(1-M_f\beta) + a(\psi - N)) > 0$.

For the PP solution, we have $\pi_2^{PP,BR} = \frac{\lambda\epsilon_2}{(1-M)(1-M_f\beta)+a(\psi-N)} \geq 0$ and therefore $\psi\pi_2^{PP,BR} > -\mu$.[30] Further:

$$\frac{\partial\pi_1^{PP,BR}}{\partial\epsilon_1} = \frac{\lambda}{(1-Mp)(1-M_f\beta p) + a(\psi - Np)} > 0$$

and $\psi\pi_1^{PP,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon^{PP,BR}$ where $\epsilon^{PP,BR}$ is defined in (A12) with $q = 1$. Therefore, PP exists if and only if $\epsilon_1 > \epsilon^{PP,BR}$, and a ZP solution always exists if $\epsilon_1 = \epsilon^{PP}$.

For the ZP solution, we have $\pi_2^{ZP,BR} = \pi_2^{PP,BR}$ and therefore $\psi\pi_2^{ZP,BR} > -\mu$. If $\chi^1 \neq 0$, then:

$$\frac{\partial\pi_1^{ZP,BR}}{\partial\epsilon_1} = \frac{\lambda}{\chi^1}$$

and $\psi\pi_1^{ZP,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon^{PP,BR}$ where $\epsilon^{PP,BR}$ is defined in (A12) with $q = 1$. Therefore if $\chi^1 > 0$ then $\epsilon^{PP,BR} \geq \epsilon_1 > \epsilon_2^{ZP,BR} = -\infty$ is necessary and sufficient for existence of the ZP solution. Otherwise, if $\chi^1 < 0$ then $\epsilon_2^{ZP,BR} = +\infty$ and $\epsilon_1 \geq \epsilon^{PP,BR}$ is necessary and sufficient for existence of the ZP solution. Note that $\delta < 0$ implies $\chi^1 > 0$. Finally, $\chi^1 = 0$ implies:[31]

$$\epsilon_2^{ZP,BR} - \epsilon^{PP,BR} = \frac{(1-p)(1 - MM_f\beta p)}{\lambda p\psi}\mu +$$

$$\frac{(1-p)(1 - MM_f\beta p)Np\psi\epsilon_2}{p\psi\left((1-M)(1-M_f\beta)p + (1-N)(1-p)(1-MM_f\beta p) + (1-Mp)(1-M_fp\beta)(\psi-1)\right)} \geq 0$$

and that a continuum of ZP solutions exist if $\epsilon_1 = \epsilon^{PP,BR}$, and no ZP solution exists if $\epsilon_1 \neq \epsilon^{PP,BR}$.

For the PZ solution, we have $\pi_2^{PZ,BR} = -\frac{\lambda\epsilon_2 + a\mu}{\delta}$. If $\delta < 0$, then $\pi_2^{PZ,BR} \geq 0$, and if $\delta > 0$, then $\psi\pi_2^{PZ,BR} = -\psi\frac{\lambda\epsilon_2+a\mu}{\delta} \leq -\psi\mu < -\mu$, since $\delta \leq a$ and $\epsilon_2 \geq 0$. If $\delta = 0$ then a PZ solution does not exist. Therefore, $\psi\pi_2^{PZ,BR} + \mu < 0$ if and only if $\delta > 0$. Further,

$$\frac{\partial\pi_1^{PZ,BR}}{\partial\epsilon_1} = \frac{\lambda}{(1-Mp)(1-M_f\beta p) + a(\psi - Np)} > 0$$

---

[30]It can be shown that $(Q + \lambda\sigma\psi I_2)^{-1}$ exists if $q = 1$.

[31]The $\chi^1 = 0$ case arises if $a = \frac{(1-Mp)(1+M_f\beta(1-p-q))+M_f(q-1)(1-M)\beta}{Np}$ and $q = 1$. To compute $\epsilon_2^{ZP,BR}$, set $a = \frac{(1-Mp)(1+M_f\beta(1-p-q))+M_f(q-1)(1-M)\beta}{Np}$ and compute $\lim_{q\to 1}\epsilon_2^{ZP,BR}$.

and $\psi\pi_1^{PZ,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon_1^{PZ,BR}$ where

$$
\begin{aligned}
\epsilon_1^{PZ,BR} &= \epsilon^{PP,BR} + \frac{a(1-p)(M(1-M_f\beta p) + aN + M_f\beta(1-M))(\lambda\psi\epsilon_2 + \mu\eta_1)}{\lambda\delta(a\psi - \delta)} \\
&\geq \epsilon^{PP,BR}
\end{aligned}
$$

and $\epsilon^{PP,BR}$ is defined in (A12) with $q = 1$. It follows that PZ exists if and only if $\delta > 0$ and $\epsilon_1 > \epsilon_1^{PZ,BR} \geq \epsilon^{PP,BR}$.

For the ZZ solution, we have $\pi_2^{ZZ,BR} = \pi_2^{PZ,BR}$, and therefore $\psi\pi_2^{ZZ,BR} + \mu \leq 0$ if and only if $\delta > 0$. Furthermore, if $\chi^1 \neq 0$ then

$$
\frac{\partial\pi_1^{ZZ,BR}}{\partial\epsilon_1} = \frac{\lambda}{\chi^1}
$$

and $\psi\pi_1^{ZZ,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon_1^{ZZ,BR} = \epsilon_1^{PZ,BR} \geq \epsilon^{PP,BR}$ and $\epsilon^{PP,BR}$ is defined in (A12) with $q = 1$. Therefore if $\chi^1 > 0$ and $\delta > 0$ then $\epsilon_1^{ZZ,BR} \geq \epsilon_1 > \epsilon_2^{ZP,BR} = -\infty$ is necessary and sufficient for existence of the ZZ solution. Otherwise, if $\chi^1 < 0$ then $\epsilon_2^{ZP,BR} = +\infty$ and $\epsilon_1 \geq \epsilon_1^{ZZ,BR} \geq \epsilon^{PP,BR}$ is necessary and sufficient for existence of the ZZ solution. If $\chi^1 = 0$ and $\delta > 0$ then $\epsilon_2^{ZP,BR} - \epsilon^{PP,BR} \geq 0$ as shown above and a continuum of ZZ solutions exist if and only if

$$
\begin{aligned}
\epsilon_1 &= \epsilon^{PP,BR} + \epsilon_2 + \frac{(1-Mp)(1-M_fp\beta)\mu}{\lambda Np} + \\
&\quad \frac{\epsilon_2(1-p)N(1-M_fMp\beta)}{(1-p)(1-MM_fp\beta)(\psi-N) + (1-M)p(1-M_f\beta)\psi} \geq \epsilon^{PP,BR}
\end{aligned}
$$

We conclude that a BRE exists if and only if

$$
\epsilon_1 \geq \bar{\epsilon}_{BR} := \begin{cases} \min\left\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\right\}, & \text{if } \delta \geq 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \tag{A15}
$$

where $\epsilon^{PP,BR}$ and $\epsilon_2^{ZP,BR}$ are defined in (A12) and (A13), respectively, with $q = 1$.

A.5 Proof of Proposition 5

Suppose $\delta = (M-1)(1-M_f\beta) + aN < 0$, which implies $det(Q) = den^{ZZ,BR} = -\delta(-\delta + (2-q-p)(M+aN+M_f\beta(1-(p+q)M))) > 0$ and $det(Q + \lambda\sigma\psi e_2 e_2') = den^{ZP,BR} = \delta^2 - \delta(a\psi + (2-p-q)(M+aN+M_f\beta(1-M(p+q)))) + (1-p)a\psi(M+aN+M_f\beta(1-M(p+q))) > 0$. First, $\delta < 0$ implies $num_1^{PZ,BR} = ((1-q)(M+aN+M_f\beta(1-M(p+q))) - \delta) > 0$ and hence no PZ solution exists since $num_1^{PZ,BR} < 0$ is necessary for existence of PZ solution, as demonstrated in the proof of Proposition 4.

Also from the proof of Proposition 4, $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} > 0$ if and only if $\delta < 0$, and therefore a unique ZZ solution exists if and only if $q < 1$ and $\epsilon_1 \leq \epsilon_2^{ZP,BR}$. From

the proof of Proposition 4, we have

$$
\epsilon^{PP,BR} - \epsilon_2^{ZP,BR} = \frac{(\mu((1-M)(1-M_f\beta)+a(\psi-N))+\lambda\epsilon_2\psi)den^{ZP,BR}}{\lambda\epsilon_{\Delta ZP,BR}} > 0
$$

$$
\epsilon_{\Delta ZP,BR} = (1-q)num_1^{ZP,BR}(M+aN+M_f\beta(1-M(p+q))) > 0
$$

$$
num_1^{ZP,BR} = \psi(a(\psi-Nq)+\beta M_f(M(p(q-1)+q(q-1)+1)-q)-Mq+1) > 0
$$

if $q < 1$ where the first inequality follows from $den^{ZP,BR} > 0$ if $\delta < 0$. If $q = 1$ then $\epsilon_2^{ZP,BR} = -\infty < \epsilon^{PP,BR}$. Therefore, $\epsilon^{PP,BR} > \epsilon_2^{ZP,BR}$ which implies that a unique ZP solution exists if and only if $\epsilon^{PP,BR} \geq \epsilon_1 > \epsilon_2^{ZP,BR}$ since $den^{ZP,BR} > 0$ from above. Define $\epsilon^{ZP,BR} \equiv \epsilon_2^{ZP,BR}$. We conclude that the PP solution is the unique BRE when $\epsilon_1 > \epsilon^{PP,BR}$, the ZP solution is the unique BRE when $\epsilon^{PP,BR} \geq \epsilon_1 > \epsilon^{ZP,BR}$. Otherwise, the ZZ solution is the unique solution if $q < 1$ and $\epsilon_1 \leq \epsilon^{ZP,BR}$. If $\delta \geq 0$ then by Proposition 4 there exist p, q, ϵ_1 and $\epsilon_2 \geq 0$ for which there are no solutions or multiple solutions.

Alternatively, one can show that $(M-1)(1-M_f\beta)+\lambda\sigma N < 0$ ensures completeness and coherence using techniques developed by AM.[32]

A.6 Endogenous Bounded Rationality

Following Moberly (2022), this section models bounded rationality as an optimal choice by agents who face a cost of paying attention in the spirit of Gabaix (2020). To that end, consider the modified version of (1)-(3)

$$
x_t = M_{\epsilon_t}E_tx_{t+1} - \sigma(i_t - E_t\pi_{t+1}) + \epsilon_t \tag{A16}
$$

$$
\pi_t = \lambda x_t + M_{f,\epsilon_t}\beta E_t\pi_{t+1} \tag{A17}
$$

$$
i_t = \max\{\psi\pi_t, -\mu\} \tag{A18}
$$

where ϵ_t, p, q, etc., are defined in the main text, and $p < 1$, $q = 1$, $\epsilon_1 = \epsilon < 0$ and $\epsilon_2 = 0$. In line with section 3.1 of Moberly (2022) we assume that $0 \leq M_{f,\epsilon_t} \leq 1$ and $0 \leq M_{\epsilon_t} \leq 1$ can switch values when ϵ_t changes: $(M_{f,\epsilon_t}, M_{\epsilon_t}) = (M_{f,i}, M_i)$ if and only if $\epsilon_t = \epsilon_i$, for $i = 1, 2$.

In a moment, we describe how $M_{f,\epsilon_t}, M_{\epsilon_t}$ arise endogenously following the approach of Moberly (2022). First, we establish the set of MSV solutions. For given $M_{f,\epsilon_t}, M_{\epsilon_t}$, there are four possible types of MSV solutions: PP, ZP, PZ and ZZ solutions. Let $l \in \{PP, ZP, PZ, ZZ\}$ denote a MSV solution. Then in the ZP and PP equilibria, the "high state" ($\epsilon_t = \epsilon_2$) equilibrium outcomes are given by:

$$
x_2^l = \pi_2^l = i_2^l = r_2^l = mc_2^l = 0
$$

where $r_2^l = i_2^l - \pi_2^l$ is the ex ante real interest rate and $mc^l = (\phi + \sigma)y_2^l$ is marginal cost.[33]

---

[32]Results available on request.
[33]See Moberly (2022) for the microfoundations.

For the PZ and ZZ solutions, the "high state" outcomes are given by:

$$x_2^l = \left(1 - M_2 - \frac{\sigma\lambda}{1 - \beta M_{f,2}}\right)^{-1}\sigma\mu$$

$$\pi_2^l = \frac{\lambda}{1 - \beta M_{f,2}}y_2^l$$

$$mc_2^l = (\phi + \sigma)y_2^l$$

$$r_2^l = -\mu - \pi_2^l$$

The low state PP equilibrium is characterized by:

$$x_1^{PP} = \left(1 - pM_1 + \frac{(\psi - p)\sigma\lambda}{1 - p\beta M_{f,1}}\right)^{-1}\epsilon_1$$

$$\pi_1^{PP} = \frac{\lambda}{1 - p\beta M_{f,1}}y_2^{PP}$$

$$mc_1^{PP} = (\phi + \sigma)y_1^{PP}$$

$$r_1^{PP} = \psi\pi_1^{PP} - p\pi_1^{PP}$$

Similarly, the low state ZP equilibrium is characterized by:

$$x_1^{ZP} = \left(1 - pM_1 - \frac{p\sigma\lambda}{1 - p\beta M_{f,1}}\right)^{-1}(\sigma\mu + \epsilon_1)$$

$$\pi_1^{ZP} = \frac{\lambda}{1 - p\beta M_{f,1}}y_1^{ZP}$$

$$mc_1^{ZP} = (\phi + \sigma)y_1^{ZP}$$

$$r_1^{ZP} = -\mu - p\pi_1^{ZP}$$

The low state PZ equilibrium is given by:

$$x_1^{PZ} = \left(1 - pM_1 + \frac{(\psi - p)\sigma\lambda}{1 - p\beta M_{f,1}}\right)^{-1}\left((1 - p)\left(M_1 y_2^{PZ} + \left(\frac{(p - \psi)\sigma M_{f,1}\beta}{1 - p\beta M_{f,1}} + \sigma\right)\pi_2^{PZ}\right) + \epsilon_1\right)$$

$$\pi_1^{PZ} = \frac{\lambda y_1^{PZ} + M_{f,1}\beta(1 - p)\pi_2^{PZ}}{1 - p\beta M_{f,1}}$$

$$mc_1^{PZ} = (\phi + \sigma)y_1^{PZ}$$

$$r_1^{PZ} = \psi\pi_1^{PZ} - p\pi_1^{PZ} - (1 - p)\pi_2^{PZ}$$

Finally, low state ZZ equilibrium is given by:

$$
\begin{aligned}
x_1^{ZZ} &= \left(1 - pM_1 - \frac{p\sigma\lambda}{1 - p\beta M_{f,1}}\right)^{-1} \left((1-p)\left(M_1 y_2^{PZ} + \left(\frac{p\sigma M_{f,1}\beta}{1 - p\beta M_{f,1}} + \sigma\right)\pi_2^{PZ}\right) + \epsilon_1 + \sigma\mu\right) \\
\pi_1^{ZZ} &= \frac{\lambda y_1^{ZZ} + M_{f,1}\beta(1-p)\pi_2^{PZ}}{1 - p\beta M_{f,1}} \\
mc_1^{ZZ} &= (\phi + \sigma)y_1^{ZZ} \\
r_1^{ZZ} &= -\mu - p\pi_1^{ZZ} - (1-p)\pi_2^{ZZ}
\end{aligned}
$$

Following Moberly (2022) we model the choice of attention of each household and firm. In our framework, households and firms need to balance the loss of not paying attention with the cost of paying attention. First, consider the household problem. The household consumption function in the high state is given by.[34]

$$
\begin{aligned}
c_t^l &= \sum_{h\geq 0} (\beta m_2)^h ((1-\beta)x_2^l - \beta r_2^l) \\
&= \frac{X_2}{1 - \beta m_2}
\end{aligned}
$$

where $M_2 = m_2$ in equilibrium. Following, Moberly (2022) we can show that the attention decision in the high state *when $X_2 \neq 0$* (i.e. the optimal m_2) depends chiefly on the following quantity:

$$
E\left[\left(\frac{\partial c_t^l}{\partial m_2}\right)^2\right] = \frac{(\beta X_2)^2}{(1 - \beta m_{d,2})^4}
$$

where the derivative is evaluated at some default level of attention $m_{d,2} \in [0,1]$. The households' optimal attention parameter in the high state of solution l, $\bar{m}_2^l$, for given M_1, $M_{f,1}$, M_2, $M_{f,2}$ is

$$
\bar{m}_2^l = \max\left(m_{d,2}, 1 - \frac{\xi_c^2}{E\left[\left(\frac{\partial c_t^l}{\partial m_2}\right)^2\right]}\right)
$$

where the parameter ξ_c is the scale-free cost of attention. Having pinned down $M_2 = \bar{m}_2^l$, we can characterize optimal m_1. In the low state, consumption is given by:

$$
\begin{aligned}
c_t^l &= E_t\left\{\sum_{h\geq 0} \beta^h \Pi_{k=1}^h m_{t+k}((1-\beta)x_{t+h} - \beta(r_{t+h} - \epsilon_{t+h}))\right\} \\
&= E_t\left\{\sum_{h\geq 0} \beta^h \Pi_{k=1}^h m_{t+k} X_{t+h}\right\}
\end{aligned}
$$

---

[34]For convenience we set $\sigma = 1$.

where $m_{t+k} = m_1$ and $X_{t+k} = X_1$ if $\epsilon_{t+k} = \epsilon_1$, otherwise $m_{t+k} = m_2$ and $X_{t+k} = X_2$. For $h \geq 1$ we have

$$
\begin{aligned}
E_t \beta^h \Pi_{k=1}^h m_{t+k} X_{t+h} &= \beta^h \left((m_1 p)^h X_1 + \left((m_1 p)^{h-1} m_2 + (m_1 p)^{h-2} (m_2)^2 + \ldots + m_2^h \right) (1-p) X_2 \right) \\
&= \beta^h \left((m_1 p)^h X_1 + (1-p) \frac{m_2^{h+1} - m_2 (m_1 p)^h}{m_2 - m_1 p} X_2 \right)
\end{aligned}
$$

Substituting the last equation into the equation for c_t and rearranging yields:

$$
c_t^l = \frac{1}{1 - \beta m_1 p} X_1 + \frac{(1-p)m_2}{m_2 - m_1 p} \left(\frac{\beta m_2}{1 - \beta m_2} - \frac{\beta m_1 p}{1 - \beta m_1 p} \right) X_2
$$

Note that

$$
\frac{(1-p)m_2}{m_2 - m_1 p} \left(\frac{\beta m_2}{1 - \beta m_2} - \frac{\beta m_1 p}{1 - \beta m_1 p} \right) = \frac{(1-p)\beta m_2}{(1 - \beta m_2)(1 - p\beta m_1)}
$$

Hence, the equation for c_t can be expressed as

$$
\begin{aligned}
c_t^l &= \frac{1}{1 - \beta m_1 p} X_1 + \frac{(1-p)\beta m_2}{(1 - \beta m_2)(1 - p\beta m_1)} X_2 \\
&= \frac{1}{1 - \beta m_1 p} \left(X_1 + \frac{(1-p)\beta m_2}{1 - \beta m_2} X_2 \right)
\end{aligned}
$$

In equilibrium, $M_1 = m_1$ and $M_2 = m_2$. Moberly (2022) shows that the attention decision in the low state (i.e. the optimal m_1) depends chiefly on the following quantity:

$$
E \left[\left(\frac{\partial c_t^l}{\partial m_1} \right)^2 \right] = \frac{(\beta p)^2 \left(X_1 (1 - \bar{m}_2^l \beta) + \bar{m}_2^l (1-p) \beta X_2 \right)^2}{(1 - \beta p m_{d,1})^4 (1 - \bar{m}_2^l \beta)^2}
$$

where the derivative is evaluated at some default level of attention $m_{d,1} \in [0,1]$ and $m_2 = \bar{m}_2^l$ is assumed. The households' optimal attention parameter in solution l, $\bar{m}_1^l$, for given M_1, M_2 is

$$
\bar{m}_1^l = \max \left(m_{d,1}, 1 - \frac{\xi_c^2}{E \left[\left(\frac{\partial c_t^l}{\partial m_1} \right)^2 \right]} \right)
$$

where again the parameter ξ_c is the scale-free cost of attention.

Now consider the firm problem.[35] The firm pricing function in the high state of solution l is given by

---

[35] We assume that firms have their own cognitive discount factor (see Gabaix (2020), footnote 13). We arrive at qualitatively similar results if we assume that households and firms have the same congitive discount factor.

$$q_t^l = (1-\beta\theta)E_t\left\{\sum_{h\geq 0}(\beta\theta m_{f,2})^h\left(\pi_{t+1}+\ldots\pi_{t+h}+mc_{t+h}\right)\right\}$$

$$= (1-\beta\theta)\sum_{h\geq 0}(\beta\theta m_{f,2})^h mc_2^l + (1-\beta\theta)\sum_{h\geq 1}(\beta\theta m_{f,2})^h h\pi_2^l$$

$$= \frac{(1-\beta\theta)}{1-\beta\theta m_{f,2}}mc_2^l + \frac{(1-\beta\theta)\beta\theta m_{f,2}}{(1-\beta\theta m_{f,2})^2}\pi_2^l$$

where q is the relative price (see Gabaix (2020) and $M_{f,2} = m_{f,2}\left(\theta + (1-\theta)\frac{1-\beta\theta}{1-\beta\theta m_{f,2}}\right)$ in equilibrium. The relevant quantity for attention choice is $E\left[\left(\frac{\partial q_t^l}{\partial m_{f,2}}\right)^2\right]$ where

$$\frac{\partial q_t^l}{\partial m_{f,2}} = \frac{\beta\theta(1-\beta\theta)(mc_2^l(1-\beta\theta m_{f,2}) + \pi_2^l(1+m_{f,2}\beta\theta))}{(1-\beta\theta m_{f,2})^3}$$

and the derivative is evaluated at some default level of attention $m_{d,f,2} \in [0,1]$. The firm's optimal attention parameter in solution l, $\bar{m}_{f,1}^l$, for given $M_{f,2}$ is

$$\bar{m}_{f,2}^l = \max\left(m_{d,f,2}, 1 - \frac{\xi_f^2}{E\left[\left(\frac{\partial q_t^l}{\partial m_{f,2}}\right)^2\right]}\right)$$

where the parameter ξ_f is the scale-free cost of attention. Now consider the low state pricing function:

$$q_t^l = (1-\beta\theta)E_t\left\{\sum_{h\geq 0}(\beta\theta)^h\Pi_{k=1}^h m_{f,t+k}\left(\pi_{t+1}+\ldots\pi_{t+h}+mc_{t+h}\right)\right\}$$

$$= (1-\beta\theta)E_t\left\{\sum_{h\geq 0}(\beta\theta)^h\Pi_{k=1}^h m_{f,t+k}mc_{t+h}\right\} +$$

$$(1-\beta\theta)E_t\left\{\sum_{h\geq 1}(\beta\theta)^h\Pi_{k=1}^h m_{f,t+k}\left(\pi_{t+1}+\ldots+\pi_{t+h}\right)\right\}$$

$$= \frac{(1-\beta\theta)}{1-p\beta\theta m_{f,1}}mc_1 + \frac{(1-\beta\theta)(1-p)\beta\theta m_{f,2}}{(1-\beta\theta m_{f,2})(1-p\beta\theta m_{f,1})}mc_2$$

$$+ (1-\beta\theta)E_t\left\{\sum_{h\geq 1}(\beta\theta)^h\Pi_{k=1}^h m_{f,t+k}\left(\pi_{t+1}+\ldots+\pi_{t+h}\right)\right\}$$

where

$$E_t \sum_{h \geq 0} (\beta\theta)^h \Pi^h_{k=1} m_{f,t+k} mc_{t+h} = \frac{1}{1 - p\beta\theta m_{f,1}} mc_1 + \frac{(1-p)\beta\theta m_{f,2}}{(1 - \beta\theta m_{f,2})(1 - p\beta\theta m_{f,1})} mc_2$$

can be derived by following steps used to derive the consumption function. Now focus on the term involving expected future inflation.

$$E_t \left\{ \sum_{h \geq 1} (\beta\theta)^h \Pi^h_{k=1} m_{f,t+k} \left(\pi_{t+1} + \ldots + \pi_{t+h} \right) \right\}$$

$$= (1-p) \sum_{h \geq 1} (\beta\theta m_{f,2})^h h\pi_2$$

$$+ \quad p(1-p) \left(\beta\theta m_{f,1}\pi_1 + \beta\theta m_{f,1} \sum_{h \geq 1} (\beta\theta m_{f,2})^h (\pi_1 + h\pi_2) \right)$$

$$+ \quad p^2(1-p) \left(\beta\theta m_{f,1}\pi_1 + (\beta\theta m_{f,1})^2 2\pi_1 + (\beta\theta m_{f,1})^2 \sum_{h \geq 1} (\beta\theta m_{f,2})^h (2\pi_1 + h\pi_2) \right)$$

$$+ \quad \ldots$$

$$= (1-p) \sum_{i \geq 0} p^i \left(\sum_{k=1}^{i} (\beta\theta m_{f,1})^k k\pi_1 + (\beta\theta m_{f,1})^i \sum_{h \geq 1} (\beta\theta m_{f,2})^h (i\pi_1 + h\pi_2) \right)$$

First, consider the term $\sum_{i \geq 0} p^i \sum_{k=1}^{i} (\beta\theta m_{f,1})^k k\pi_1$:

$$\sum_{i \geq 0} p^i \sum_{k=1}^{i} (\beta\theta m_{f,1})^k k\pi_1 = p\beta\theta m_{f,1}(1 + p + p^2 + \ldots)\pi_1$$

$$+ \quad (p\beta\theta m_{f,1})^2(1 + p + p^2 + \ldots)2\pi_1$$

$$+ \quad (p\beta\theta m_{f,1})^3(1 + p + p^2 + \ldots)3\pi_1$$

$$+ \quad \ldots$$

$$= \frac{p\beta\theta m_{f,1}}{(1-p)(1 - p\beta\theta m_{f,1})^2}\pi_1$$

where the last equality uses the fact that $\sum_{h \geq 1} a^h h = a/(1-a)^2$ if $|a| < 1$. Now consider the term:

$$\sum_{i \geq 0} (p\beta\theta m_{f,1})^i \sum_{h \geq 1} (\beta\theta m_{f,2})^h i\pi_1 = \frac{\beta\theta m_{f,2}}{1 - \beta\theta m_{f,2}} \sum_{i \geq 0} (p\beta\theta m_{f,1})^i i\pi_1$$

$$= \frac{\beta\theta m_{f,2}}{1 - \beta\theta m_{f,2}} \sum_{i \geq 1} (p\beta\theta m_{f,1})^i i\pi_1 = \frac{p(\beta\theta)^2 m_{f,2} m_{f,1}}{(1 - p\beta\theta m_{f,1})^2(1 - \beta\theta m_{f,2})}\pi_1$$

Finally, consider the term:

$$\sum_{i\geq 0}(p\beta\theta m_{f,1})^i \sum_{h\geq 1}(\beta\theta m_{f,2})^h h\pi_2 = \sum_{i\geq 0}(p\beta\theta m_{f,1})^i \frac{\beta\theta m_{f,2}}{(1-\beta\theta m_{f,2})^2}\pi_2 = \frac{\beta\theta m_{f,2}}{(1-p\beta\theta m_{f,1})(1-\beta\theta m_{f,2})^2}\pi_2$$

Therefore:

$$E_t\left\{\sum_{h\geq 1}(\beta\theta\Pi_{k=1}^h m_{f,t+k})^h\left(\pi_{t+1}+\ldots+\pi_{t+h}\right)\right\}$$

$$= (1-p)\left(\frac{p\beta\theta m_{f,1}}{(1-p)(1-p\beta\theta m_{f,1})^2}+\frac{p(\beta\theta)^2 m_{f,1}m_{f,2}}{(1-p\beta\theta m_{f,1})^2(1-\beta\theta m_{f,2})}\right)\pi_1$$

$$+ \frac{(1-p)\beta\theta m_{f,2}}{(1-p\beta\theta m_{f,1})(1-\beta\theta m_{f,2})^2}\pi_2$$

Putting things together:

$$q_t = \frac{(1-\beta\theta)}{1-\beta\theta m_{f,1}p}mc_1 + \frac{(1-\beta\theta)(1-p)\beta\theta m_{f,2}}{(1-\beta\theta m_{f,2})(1-p\beta\theta m_{f,1})}mc_2$$

$$+ \left(\frac{(1-\beta\theta)p\beta\theta m_{f,1}}{(1-p\beta\theta m_{f,1})^2}+\frac{(1-\beta\theta)(1-p)p(\beta\theta)^2 m_{f,1}m_{f,2}}{(1-p\beta\theta m_{f,1})^2(1-\beta\theta m_{f,2})}\right)\pi_1$$

$$+ \frac{(1-\beta\theta)(1-p)\beta\theta m_{f,2}}{(1-p\beta\theta m_{f,1})(1-\beta\theta m_{f,2})^2}\pi_2$$

The relevant quantity for attention choice is $E\left[\left(\frac{\partial q_t^l}{\partial m_{f,1}}\right)^2\right]$ where the derivative is evaluated at some default level of attention $m_{d,f,1}\in[0,1]$. For brevity, the expression for $E\left[\left(\frac{\partial q_t^l}{\partial m_{f,1}}\right)^2\right]$ is omitted from these notes. The firm's optimal attention parameter in solution l, $\bar{m}_{f,1}^l$, given $\bar{m}_{f,2}^l$ is

$$\bar{m}_{f,1}^l = \max\left(m_{d,f,1}, 1-\frac{\xi_f^2}{E\left[\left(\frac{\partial q_t^l}{\partial m_{f,1}}\right)^2\right]}\right)$$

We define an endogenous BRE as follows.

Definition 8 *An **endogenous BRE** is a tuple,* $\mathbf{m}^* = (m_1^*, m_{f,1}^*, m_2^*, m_{f,2}^*)$, *and a vector of endogenous variables,* $\mathbf{x}^* = (y_1, y_2, \pi_1, \pi_2, i_1, i_2)$, *such that*

1. $m_1^* = \bar{m}_1$, $m_{f,1}^* = \bar{m}_{f,1}$, $m_2^* = \bar{m}_2$, *and* $m_{f,2}^* = \bar{m}_{f,2}$ *taking* $M_1 = m_1^*$, $M_2 = m_2^*$, $M_{f,1} = m_{f,1}^*\left(\theta+(1-\theta)\frac{1-\beta\theta}{1-\beta\theta m_{f,1}^*}\right)$, $M_{f,2} = m_{f,2}^*\left(\theta+(1-\theta)\frac{1-\beta\theta}{1-\beta\theta m_{f,2}^*}\right)$, *and* $\mathbf{x}^*$ *as given.*

2. $\mathbf{x}^*$ *solves (A16)-(A18) given* $M_1 = m_1^*$, $M_2 = m_2^*$, $M_{f,1} = m_{f,1}^*\left(\theta+(1-\theta)\frac{1-\beta\theta}{1-\beta\theta m_{f,1}^*}\right)$

$$\text{and } M_{f,2} = m^*_{f,2}\left(\theta + (1-\theta)\frac{1-\beta\theta}{1-\beta\theta m^*_{f,2}}\right).$$

An endogenous BRE is a BRE in which the agents' discount factors are optimally chosen taking the economy-wide discount factors as given.

We solve for endogenous BRE numerically using the calibration: $\beta = 0.99$, $\sigma = 1$, $\psi = 2$, $\lambda = 0.02$, $p = 0.9$, $q = 1$, $\xi_c = \xi_f = 0.01$ and $m_{d,1} = m_{d,2} = m_{d,f,1} = m_{d,f,2} = 0.7$. The scale-free attention cost parameters are set to 0.01 which means that households/firms pay attention to variables that make a 1% difference for decisions on average. The relatively low value of the default discount factor means that agents initially discount the future heavily. For each calibration of ϵ_1 we solve for endogenous BRE.

For the ZP and PP cases, we note that since $x^l_2 = \pi^l_2 = r^l_1 = mc^l_2 = 0$ for $l = ZP, PP$, the optimal discount factors, $\bar{m}^l_2$ and $\bar{m}^l_{f,2}$ are undetermined in the ZP and PP solutions. We therefore assume that agents set $m_2 = \bar{m}^l_1$ and $m_{f,2} = \bar{m}^l_{f,1}$ in the ZP and PP equilibria. In other words, agents are assumed to choose the same discount factor in both states to solve the low state optimization problem. Characterizing the endogenous BRE in these cases boils down to solving for $\bar{m}^l_1$ and $\bar{m}^l_{f,1}$. We find that neither the ZP nor the PP solution exists if $\epsilon_1 < -0.014$. Intuitively, a large (negative) shock necessitates greater endogenous attention, which in turn implies high endogenous BRE values of the discount parameters and hence no solution. For this calibration, no REE exists for sufficiently negative values of ϵ_1, but an RPE exists for any value of ϵ_1.

In general, a sufficiently negative value of ϵ_1 implies no solution; support restrictions on the shock are needed to generate endogenous BRE.

Remark: *For any standard calibration of the model, the ZP or PP solution only exists if $\epsilon_1 > \bar{\epsilon}_{EBRE}$ for some $\bar{\epsilon}_{EBRE} < 0$.*

Intuitively, $\epsilon_1 = -\infty$ implies $E\left[\left(\frac{\partial c^l_t}{\partial m_1}\right)^2\right] = \infty$ for $l = ZP, PP$ and for any m_1. Therefore $M_1 = m^*_1 = 1$ is the only candidate endogenous BRE value of m^*_1. However, if $M_1 = 1$ then the model is incoherent for high values of $|\epsilon_1|$.

For the PZ and ZZ cases, we find: $\bar{m}^l_2 = m^*_2 = 0.8977$ and $\bar{m}^l_{f,2} = m^*_{f,2} = 0.9808$ which implies $M_{f,2} = 0.9677$ for $l = PZ, ZZ$. We then solve for the remaining low state discount factors for different values of ϵ_1. As in the ZP and PP cases, we find that the PZ and ZZ solutions may not exist for $\epsilon_1 < -0.14$. Therefore, the support of the shock must be restricted for an endogenous BRE to exist.

A.7 Proof of Proposition 6 and E-stability of BR-RPE

A.7.1 Proof of Proposition 6

The proof of Proposition 6 is a straightforward extension of the proof of Proposition 4. Define $\bar{q} = Pr(s_t = 2) = (1-p)/(2-p-q)$. The regime-specific levels of inflation in BR-RPE i, $\hat{\pi}^i = (\pi^i_1, \pi^i_2)'$, are given by fixed point restrictions that have the same basic form as the BRE fixed point restrictions except we replace q with $\bar{q}$ and p with $1 - \bar{q}$. Therefore, BR-RPE will

exist if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{BR,RPE} := \begin{cases} \min\left\{\epsilon^{PP,BR,RPE}, \epsilon_2^{ZP,BR,RPE}\right\}, & \text{if } \delta \geq 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \quad \text{(A19)}$$

where $\delta = (M-1)(1-M_f\beta) + \lambda\sigma N$, and $\epsilon^{PP,BR,RPE}$ and $\epsilon_2^{ZP,BR,RPE}$ are defined in (A12) and (A13), respectively, assuming $p = 1 - \bar{q}$, and $q = \bar{q}$. In the special case $q = 1$ (which implies $\bar{q} = 1$), we have $\bar{\epsilon}_{BR,RPE} = -\infty$ for any δ, as the PP solution exists if and only if $\epsilon_1 > -\mu(1+\lambda\sigma\psi)(\lambda\psi)^{-1} + (M(1-M_f\beta)+M_f\beta+\lambda\sigma N)((M-1)(1-M_f\beta)+\lambda\sigma(N-\psi))^{-1}\epsilon_2 = \epsilon^{PP,BR,RPE}$ and the ZP exists if and only if $\epsilon_1 \leq \epsilon^{PP,BR,RPE}$.

For $q < 1$, one can show:

$$\begin{aligned} \epsilon^{PP,BR} - \epsilon^{PP,BR,RPE} &= \Xi_{PP}^B(1-p-q) \\ \epsilon^{ZP,BR} - \epsilon^{ZP,BR,RPE} &= \Xi_{ZP}^B(1-p-q) \end{aligned}$$

where

$$\begin{aligned} \Xi_{PP}^B &= \frac{(p-1)(\eta_6 + MM_f\beta)\eta_7(\lambda\epsilon_2\psi + \mu\eta_7)}{\psi(\eta_7 + (1-q)\eta_6)((1-p)(\lambda\sigma - \delta) + ((1+\lambda\sigma)(1-q) + \lambda\sigma(2-q-p)(\psi-1)))} \\ \Xi_{ZP}^B &= \frac{\delta(\eta_6 + MM_f\beta)(\lambda\epsilon_2\psi + \mu\eta_7)}{(q-1)(M + \lambda\sigma N + M_f\beta(1-M))\eta_6\psi} \end{aligned}$$

.

where $\eta_6 = M(1 - M_f\beta p) + \lambda\sigma N + M_f\beta(1 - qM) > 0$, and $\eta_7 = (a(\psi - N) + (1-M)(1 - M_f\beta)) > 0$. Since $\delta \leq \lambda\sigma$, it is straightforward to show that $\Xi_{PP}^B \leq 0$. Further, if $\delta \geq 0$ then $\Xi_{ZP}^B \leq 0$. It follows that $\bar{\epsilon}_{BR} \geq \bar{\epsilon}_{BR,RPE}$ if $\delta \geq 0$ and $p + q - 1 \geq 0$ or $\delta < 0$.

A.7.2   E-stability of BR-RPE

Analogous to the RPE case, there is a unique E-stable BR-RPE.

Proposition 11 *Consider (1)-(3) and assume $\epsilon_2 \geq 0$. If $\epsilon_1 > \bar{\epsilon}_{BR,RPE}$, then:*

   i. There is a unique E-stable BR-RPE.

   ii. The E-stable BR-RPE is either the PP BR-RPE or the ZP BR-RPE if $(M-1)(1-M_f\beta) + \lambda\sigma N \geq 0$.

   iii. The E-stable BR-RPE is the unique BR-RPE if $(M-1)(1-M_f\beta) + \lambda\sigma N < 0$.

Proof. To assess E-stability of each BR-RPE, we express the BR-RPE unconditional mean of inflation and output as a function of agents' expectations, Y^e:

$$\begin{aligned} \bar{Y}^{PP}(Y^e) &= \hat{A}_P Y^e + \bar{\Gamma}^{PP}, \\ \bar{Y}^{ZP}(Y^e) &= \left(\bar{q}\hat{A}_P + (1-\bar{q})\hat{A}_Z\right)Y^e + \bar{\Gamma}^{ZP}, \\ \bar{Y}^{PZ}(Y^e) &= \left((1-\bar{q})\hat{A}_P + \bar{q}\hat{A}_Z\right)Y^e + \bar{\Gamma}^{PZ}, \\ \bar{Y}^{ZZ}(Y^e) &= \hat{A}_Z Y^e + \bar{\Gamma}^{ZZ}, \end{aligned}$$

where $\bar{\Gamma}^i$ collect terms that do not depend on beliefs, Y^e, and

$$\hat{A}_P = \begin{pmatrix} \frac{M}{1+\lambda\sigma\psi} & \frac{N\sigma - M_f\beta\sigma\psi}{1+\lambda\sigma\psi} \\ \frac{M\lambda}{1+\lambda\sigma\psi} & \frac{M_f\beta+N\lambda\sigma}{1+\lambda\sigma\psi} \end{pmatrix} \qquad \hat{A}_Z = \begin{pmatrix} M & N\sigma \\ M\lambda & M_f\beta + N\lambda\sigma \end{pmatrix}$$

It immediately follows that

$$\begin{aligned}
DT_{\bar{Y}PP} &= \hat{A}_P - I, \\
DT_{\bar{Y}ZP} &= \bar{q}\hat{A}_P + (1-\bar{q})\hat{A}_Z - I, \\
DT_{\bar{Y}PZ} &= (1-\bar{q})\hat{A}_P + \bar{q}\hat{A}_Z - I, \\
DT_{\bar{Y}ZZ} &= \hat{A}_Z - I.
\end{aligned}$$

Case $\delta = (M-1)(1-M_f\beta) + N\sigma\lambda < 0$. It is straightforward to show that the real parts of the eigenvalues of $DT_{\bar{Y}PP}$ and $DT_{\bar{Y}ZZ}$ are negative if $\delta < 0$. Therefore, the PP BR-RPE and ZZ BR-RPE are E-stable if they exist. The ZP RPE is E-stable if and only if

$$\begin{aligned}
tr(DT_{\bar{Y}ZP}) &= \delta + MM_f\beta - 1 - \bar{q}\frac{\lambda\sigma(M+M_f\beta+N\lambda\sigma)\psi}{1+\lambda\sigma\psi} < 0 \\
det(DT_{\bar{Y}ZP})) &= \frac{-\delta(1+\lambda\sigma\psi)}{1+\lambda\sigma\psi} + \frac{\lambda\sigma\psi\bar{q}(\delta+1)}{1+\lambda\sigma\psi} > 0
\end{aligned}$$

where $tr(B)$ denotes the trace of matrix B. Because $-1 < \delta$, the ZP BR-RPE is E-stable in the case $\delta < 0$. Further, this holds for any $\bar{q}$, and therefore the PZ BR-RPE is E-stable if it exists, as $tr(DT_{\bar{Y}PZ}), det(DT_{\bar{Y}PZ})$ have the same form as $tr(DT_{\bar{Y}ZP}), det(DT_{\bar{Y}ZP})$ with $\bar{q}$ replaced by $1 - \bar{q}$.

By the proof of Proposition 5 (setting $q = \bar{q}$ and $p = 1 - \bar{q}$) there is a unique BR-RPE if $\delta < 0$. Therefore, there is a unique E-stable BR-RPE if $\delta < 0$.

Case $\delta \geq 0$. It is straightforward to show that the real parts of the eigenvalues of $DT_{\bar{Y}PP}$ are negative and the real part of an eigenvalue of $DT_{\bar{Y}ZZ}$ is non-negative if $\delta \geq 0$. Therefore, the PP BR-RPE is E-stable and the ZZ BR-RPE is not E-stable in the case $\delta \geq 0$.

The ZP RPE is E-stable if and only if $tr(DT_{\bar{Y}ZP}) < 0 < det(DT_{\bar{Y}ZP})$, which holds if and only if $\eta_{ZP} = \bar{q}(\lambda\sigma(M+M_f\beta(1-M)+N\sigma)\psi) - (M-1+M_f\beta(1-M)+N\lambda\sigma)(1+\lambda\sigma\psi) > 0$. From the proofs of Propositions 4 and 6:

$$\epsilon^{PP,BR,RPE} - \epsilon_2^{ZP,BR,RPE} = v_b\eta_{ZP}$$

where $v_b = \frac{(\lambda\epsilon_2 + (\lambda\sigma - \delta\psi^{-1})\mu)\psi}{((1-\bar{q})(M+M_f\beta(1-M)+N\lambda\sigma)\psi((1-M_f\beta)(1-M)+\lambda\sigma(\psi-N\bar{q})+(1-\bar{q})(M_f\beta+M(1-M_f\beta))))} > 0$. Therefore, if the ZP RPE is E-stable then $\epsilon^{PP,BR,RPE} > \epsilon_2^{ZP,BR,RPE}$ and the condition for PP existence becomes $\epsilon_1 > \epsilon^{PP,BR,RPE}$ and the condition for ZP existence becomes $\epsilon^{PP,BR,RPE} \geq \epsilon_1 > \epsilon_2^{ZP,BR,RPE}$ as demonstrated in the proofs of Propositions 4 and 6.[36] Hence, if the ZP

---

[36]If $\bar{q} = 1$, the PP exists and is E-stable if and only if $\epsilon_1 > \epsilon^{PP,BR,RPE}$ and the ZP exists and is E-stable if and only if $\epsilon_1 \leq \epsilon^{PP,BR,RPE}$. The ZZ and PZ solutions cannot be E-stable if $\delta \geq 0$.

RPE exists and is E-stable then the PP solution does not exist.

Next consider the PZ solution. The PZ solution is E-stable if and only if

$$
\begin{aligned}
tr(DT_{\bar{Y}^{PZ}}) &= \frac{-2 + M + M_f\beta + N\lambda\sigma - 2\lambda\sigma\psi}{1 + \lambda\sigma\psi} - \bar{q}\frac{\lambda\sigma(M + M_f\beta + N\lambda\sigma)\psi)}{1 + \lambda\sigma\psi} < 0 \\
det(DT_{\bar{Y}^{PZ}})) &= \frac{1 - M_f\beta + M(M_f\beta - 1) + (\psi - N)\lambda\sigma}{1 + \lambda\sigma\psi} - \bar{q}\frac{\lambda\sigma\psi(\delta + 1)}{1 + \lambda\sigma\psi} > 0
\end{aligned}
$$

which holds if and only if $0 < den^{PZ,BR,RPE}$ where $den^{PZ,BR,RPE}$ is equal to $den^{PZ,BR}$ defined in the Proposition 4 proof when $q = \bar{q}$ and $p = 1 - \bar{q}$. From the proof of Proposition 6, the PZ RPE only exists in the case $\delta \geq 0$ if $den^{PZ,BR,RPE} < 0$. Hence the PZ BR-RPE is never E-stable if $\delta \geq 0$.

Therefore, the PP BR-RPE is the only E-stable BR-RPE solution when $\epsilon_1 > \epsilon^{PP,BR,RPE}$, and the ZP BR-RPE is the only E-stable BR-RPE solution when $\epsilon^{PP,BR,RPE} \geq \epsilon_1 > \epsilon_2^{ZP,BR,RPE}$. It follows that a unique E-stable BR-RPE solution exists when $\epsilon_1 > \bar{\epsilon}_{BR,RPE}$.

A.8 Forward Guidance Puzzle

Central banks have relied heavily on forward guidance (FG) in recent decades. A large literature established that promises to keep interest rates lower for longer at the ZLB can have implausibly large effects on inflation and output in standard New Keynesian environments. Moreover, a promise to cut a future interest rate has larger effects on today's inflation than the same cut in the current rate, and the effects become *unbounded* as the timing of the rate cut is pushed into the infinite future. These counterintuitive predictions are sometimes referred to as the "forward guidance puzzle" (see Gibbs and McClung (forth.)). In order to focus squarely on the implausible effects of anticipated *future* rate cuts, we consider the following canonical thought experiment: suppose at time $t = 0$, the central bank promises to (a) peg the interest rate at steady state until $t = T - 1 > 0$, then (b) peg the interest rate below steady state at $t = T$, and finally (c) set interest rates according to a policy rule (e.g. an active Taylor rule) for $t > T$. Formally we have:

$$
\begin{aligned}
x_t &= ME_t x_{t+1} - \sigma(i_t - NE_t\pi_{t+1}) & \text{(A20)} \\
\pi_t &= \lambda x_t + \beta M_f E_t \pi_{t+1} & \text{(A21)} \\
i_t &= \begin{cases} 0 \text{ for } t = 0, \ldots, T - 1 \\ \bar{i} < 0 \text{ for } t = T \\ \psi\pi_t \text{ for } t > T \end{cases} & \text{(A22)}
\end{aligned}
$$

To fix things, we define the forward guidance puzzle following Diba and Loisel (2021).

Definition (Forward Guidance Puzzle). *When the policy rate is pegged ($i_t = 0$) for $t = 0, \ldots, T - 1$, the time-0 response to inflation and output to an expected policy rate cut at time-T ($i_T = \bar{i} < 0$) goes to infinity with T (i.e. $\lim_{T \to \infty} \partial\pi_0/\partial i_T = \lim_{T \to \infty} \partial x_0/\partial i_T = -\infty$).*

Intuitively, the forward guidance puzzle emerges if the time-0 response of inflation/output

to a promise to cut interest rates at time $T > 0$ is *strictly increasing* in T. Using terminology from Farhi and Werning (2019), this "anti-horizon" effect of monetary policy implies that a 100-basis point cut in the current policy rate causes a smaller rise inflation today than a promise to cut by 100-basis points 10 years from now, which has a smaller effect today than a promise to cut 1000 years from now, and so on.

The model can be solved recursively through the method of undetermined coefficients combined with backward induction given agents' expectations about the economy after forward guidance ends. First, $\psi > 1$ implies the unique equilibrium, $Y_t = 0$ for $t > T$. This determines $E_T Y_{T+1} = 0$, which implies $Y_T = \Gamma^{br}$ where Γ^{br} is a function of $\bar{i}$ and the other model parameters. Therefore:

$$
\begin{aligned}
Y_{T-1} &= A_{br,z}\Gamma^{br} \\
Y_{T-2} &= A_{br,z}^2\Gamma^{br} \\
&\vdots \\
Y_0 &= A_{br,z}^T\Gamma^{br}
\end{aligned}
$$

where

$$
A_{br,z} = \left(\begin{array}{cc} M & \sigma N \\ M\lambda & M_f\beta + \lambda\sigma N \end{array} \right) \qquad \Gamma^{br} = \left(\begin{array}{c} -\sigma\bar{i} \\ -\lambda\sigma\bar{i} \end{array} \right)
$$

Alternatively, we can represent the solution as a VAR(1) process. Define $j = T - t$ and $Y_t = a_{T-j}$. Then a_j is given by

$$
\begin{aligned}
a_0 &= \Gamma^{br} \\
a_j &= A_{br,z}a_{j-1} \text{ for } j > 0
\end{aligned}
$$

The roots of $A_{br,z}$ are inside the unit circle if and only if $(M-1)(1-M_f\beta)+\lambda\sigma N < 0$. Hence, if $(M-1)(1-M_f\beta)+\lambda\sigma N < 0$ then $\lim_{j\to\infty} a_j = 0$ and therefore $\lim_{T\to\infty} \partial\pi_0/\partial i_T = 0$. If $(M-1)(1-M_f\beta)+\lambda\sigma N > 0$ then the roots of $A_{br,z}$ are outside the unit circle and one can show that $\lim_{T\to\infty} \partial\pi_0/\partial i_T = -\infty$. We summarize this result, which is nearly a restatement of Proposition 4 of Gabaix (2020), as a theorem.

Theorem 1 *Consider the forward guidance model (A20)-(A22).*

1. *The model does not exhibit the forward guidance puzzle if $(M-1)(1-M_f\beta)+\lambda\sigma N < 0$.*

2. *The model exhibits the forward guidance puzzle under RE ($M = M_f = N = 1$).*

The theorem demonstrates that the same condition ensuring coherence/completeness in the occasionally-binding constraint framework rules out the forward guidance puzzle. The model with full information RE is susceptible to the puzzle.

A.8.1   Adaptive Learning and Forward Guidance Puzzle

Now we consider the effects of forward guidance when agents adaptively forecast inflation and output. We study two models of adaptive learning. The first model is given by the

following system of equations:

$$x_t = \hat{E}_t x_{t+1} - \sigma(i_t - \hat{E}_t \pi_{t+1}) \tag{A23}$$

$$\pi_t = \lambda x_t + \beta \hat{E}_t \pi_{t+1} \tag{A24}$$

$$\hat{E}_t x_{t+1} = \gamma_{x,t} x_{t-1} + (1 - \gamma_{x,t})\hat{E}_{t-1} x_t \tag{A25}$$

$$\hat{E}_t \pi_{t+1} = \gamma_{\pi,t} \pi_{t-1} + (1 - \gamma_{\pi,t})\hat{E}_{t-1} \pi_t \tag{A26}$$

$$i_t = \begin{cases} 0 \text{ for } t = 0, \ldots, T-1 \\ \bar{i} < 0 \text{ for } t = T \\ \psi \pi_t \text{ for } t > T \end{cases} \tag{A27}$$

It is trivial to show that $\partial \pi_t / \partial i_T = \partial x_t / \partial i_T = 0$ for all $t < T$ in this framework with learning. Expectations are backward-looking and predetermined in each period $t < T$ and hence the "anticipated" interest rate cut has no effect on inflation and output until the shock hits the economy at $t = T$.

Proposition 12 *The adaptive learning model (A23)-(A27) does not exhibit the forward guidance puzzle and forward guidance announcements have no contemporaneous impact on the economy ($\partial \pi_0 / \partial i_T = \partial x_0 / \partial i_T = 0$ for all T).*

The last proposition is not entirely robust to the type of decision rules that learning agents are assumed to have. To see this, consider a second model that features infinite horizon decision rules and adaptive learning:

$$x_t = -\sigma i_t + \hat{E}_t \sum_{k \geq t} \beta^{k-t} \left((1-\beta)x_{k+1} + \sigma \pi_{k+1} - \sigma \beta i_{k+1} \right) \tag{A28}$$

$$\pi_t = \lambda x_t + \hat{E}_t \sum_{k \geq t} (\xi\beta)^{k-t} \left(\xi\beta\lambda x_{k+1} + (1-\xi)\beta \pi_{k+1} \right) \tag{A29}$$

$$\hat{E}_t x_{k+1} = \hat{E}_t x_{t+1} = \gamma_{x,t} x_{t-1} + (1 - \gamma_{x,t})\hat{E}_{t-1} x_t \tag{A30}$$

$$\hat{E}_t \pi_{k+1} = \hat{E}_t \pi_{t+1} = \gamma_{\pi,t} \pi_{t-1} + (1 - \gamma_{\pi,t})\hat{E}_{t-1} \pi_t \tag{A31}$$

$$i_t = \begin{cases} 0 \text{ for } t = 0, \ldots, T-1 \\ \bar{i} < 0 \text{ for } t = T \\ \psi \pi_t \text{ for } t > T \end{cases} \tag{A32}$$

where $\lambda = (1 - \xi\beta)(1 - \xi)/\xi$. Under infinite horizon learning, agents need to forecast the path of the nominal interest rate in addition to the paths of inflation and output. The following assumption about interest rate forecasts encodes the belief that the forward guidance announcement is credible:

$$\hat{E}_0 i_{k+1} = \begin{cases} 0 \text{ for } k = 0, \ldots, T-2 \\ \bar{i} \text{ for } k = T-1 \\ \gamma_{i,t} i_{t-1} + (1 - \gamma_{i,t})\hat{E}_{-1} i_0 \text{ for } k \geq T \end{cases} \tag{A33}$$

In other words, (A33) shows how learning agents might form expectations if the forward

guidance announcement (A32) is perceived as credible. On the other hand, interest rate expectations at $t = 0$ are given by

$$\hat{E}_0 i_{k+1} = \gamma_{i,t} i_{t-1} + (1 - \gamma_{i,t})\hat{E}_{-1} i_0 \tag{A34}$$

when the announcement is *not* credible. In either case, the forward guidance puzzle is absent.

Proposition 13 *Consider the infinite-horizon adaptive learning model (A28)-(A32).*

> *i. If the announcement is credible (interest rate expectations are given by (A33)) then there is no forward guidance puzzle and $\partial x_0/\partial i_T = -\sigma\beta^T$ and $\partial\pi_0/\partial i_T = -\lambda\sigma\beta^T$.*

> *ii. If the announcement is not credible (interest rate expectations are given by (A34)) then there is no forward guidance puzzle and $\partial x_0/\partial i_T = \partial\pi_0/\partial i_T = 0$ for any T.*

We refer interested readers to Eusepi et al. (2021) and Cole (2021) for more on forward guidance under infinite-horizon learning.

A.9 Proof of Proposition 7

Consider Proposition 7. To assess E-stability of an REE, we express $Y^i = (Y_1^{i\prime}, Y_2^{i\prime})'$ as a function of agents' expectations, $\tilde{Y}^e = (Y_1^{e\prime}, Y_2^{e\prime})'$:

$$
\begin{aligned}
Y^{PP}(\tilde{Y}^e) &= \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_P & qA_P \end{pmatrix} \tilde{Y}^e + \Gamma^{PP}, \\
Y^{ZP}(\tilde{Y}^e) &= \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_P & qA_P \end{pmatrix} \tilde{Y}^e + \Gamma^{ZP}, \\
Y^{PZ}(\tilde{Y}^e) &= \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_Z & qA_Z \end{pmatrix} \tilde{Y}^e + \Gamma^{PZ}, \\
Y^{PP}(\tilde{Y}^e) &= \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_Z & qA_Z \end{pmatrix} \tilde{Y}^e + \Gamma^{ZZ},
\end{aligned}
$$

where Γ^i collect terms that do not depend on beliefs, $\tilde{Y}^e$. It immediately follows that

$$
\begin{aligned}
DT_{Y^{PP}} &= K \otimes A_P - I, \\
DT_{Y^{ZP}} &= \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_P & qA_P \end{pmatrix} - I, \\
DT_{Y^{PZ}} &= \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_Z & qA_Z \end{pmatrix} - I, \\
DT_{Y^{ZZ}} &= K \otimes A_Z - I.
\end{aligned}
$$

REE i is E-stable if the real parts of the eigenvalues of DT_{Y^i} are negative for $i = PP, ZP, PZ, ZZ$. It is straightforward to show that the real parts of the eigenvalues of $DT_{Y^{PP}}$ are negative and the real part of an eigenvalue of $DT_{Y^{ZZ}}$ is positive. Therefore, the PP solution is always E-stable and the ZZ solution is always E-unstable. We now proceed to show that only one REE can be E-stable in two steps.

First, we show that the PP solution does not exist if the ZP solution is E-stable. Because DT_{YZP} is a 4 by 4 matrix the following condition is necessary for E-stability of the ZP solution:

$$Det(DT_{YZP}) = \frac{a}{1+a\psi}den^{ZP} > 0$$

where den^{ZP} is defined in the proof of Proposition 1. Therefore, E-stability of the ZP solution implies $den^{ZP} > 0$. Furthermore, since $den^{ZP} > 0$ implies $\epsilon^{PP} > \epsilon_2^{ZP}$ (see proof of Proposition 1), E-stability of the ZP solution implies $\epsilon^{PP} > \epsilon_2^{ZP}$, where $\epsilon^{PP}, \epsilon_2^{ZP}$ are defined in the proof of Proposition 1. Also from the Proposition 1 proof, if $\epsilon^{PP} > \epsilon_2^{ZP}$ then $\epsilon_1 > \epsilon^{PP}$ is necessary for existence of PP and $\epsilon_1 \leq \epsilon^{PP}$ is necessary for existence of ZP. It follows that the E-stability and existence of the ZP solution precludes existence of the PP solution.

Second, we show that the PZ solution is never E-stable. The following condition is necessary for E-stability of the PZ solution:

$$Det(DT_{YPZ}) = \frac{1}{1+a\psi}den^{PZ} > 0$$

where den^{PZ} is defined in the proof of Proposition 1. Therefore, $den^{PZ} > 0$ is necessary for the PZ solution to be E-stable. From the proof of Proposition 1, $den^{PZ} < 0$ is necessary for existence of the PZ solution. We conclude that the PZ solution can never be E-stable.

In sum, if the PP solution exists it is E-stable. If the ZP solution exists and is E-stable then the PP solution does not exist. The ZZ and PZ solutions are never E-stable.

A.10   Proof of Proposition 8

Consider (18)-(23), let Y_t^e denote the subjective forecast of Y_t implied by a given forecasting model, and assume that agents observe ϵ_t and Y_t when forecasting at time t. Furthermore, to deal with possible multiplicity of time-t temporary equilibria, i.e. a time-t solution of (1)-(3) given forecasts and ϵ_t with binding ZLB ($s_t = 0$) and a solution with slack ZLB constraint ($s_t = 1$), we simply assume that ϵ_t determines s_t. E.g. if $\epsilon_t = \epsilon_j$, $s_t = 0$ and $s_t = 1$ are both possible in temporary equilibrium, and $s_k = 0$ for some $k < t$ such that $\epsilon_k = \epsilon_j$, then we select the temporary equilibrium characterized by $s_t = 0$.

(i) First consider (18)-(22).

Case $k = 0$. If $k = 0$ and expectations are formed under PLMs (18)-(22) then Y_t^e follows a 2-state process: $Y_t^e = Y_j^e$ if $\epsilon_t = \epsilon_j$. Further, $\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y_1^e + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y_2^e$ is a 2-state process. Therefore, if $k = 0$ then $Y_j^e = Y_j$ is necessary and sufficient for the agents to have self-confirming beliefs under the PLMs (18)-(22). These self-confirming beliefs imply: $\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y_1 + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y_2$. Substituting $\hat{E}_t Y_{t+1}$ into the model and solving for Y_1 and Y_2 straightforwardly implies that Y_1, Y_2 is an REE. Hence, beliefs formed under (18)-(22) with $k = 0$ are only self-confirming if an REE exists.

Case $k = 1$. Beliefs are only self-confirming under the PLMs (18)-(22) with $k = 1$ if $Y_j^e = E(Y_t|\epsilon_{t-1} = \epsilon_j)$ for $j = 1, 2$ where E denotes the true mathematical expectation

operator. Further, $\hat{E}_t Y_{t+1}$ formed under (18)-(22) follows a 2-state process and therefore temporary equilibrium Y_t follows a 2-state process: Y_j, where Y_j is the actual equilibrium value of Y given Y_j^e and $\epsilon_t = \epsilon_j$ for $j = 1, 2$. It follows that beliefs are self-confirming if and only if $E(Y_t | \epsilon_{t-1} = \epsilon_1) = pY_1 + (1-p)Y_2$ and $E(Y_t | \epsilon_{t-1} = \epsilon_2) = (1-q)Y_1 + qY_2$. Therefore, if agents have self-confirming beliefs under PLMs (18)-(22) with $k = 1$ then $\hat{E}_t Y_{t+1} = Y_{t+1}^e = pY_1 + (1-p)Y_2$ if $\epsilon_t = \epsilon_1$ and $\hat{E}_t Y_{t+1} = Y_{t+1}^e = (1-q)Y_1 + qY_2$ otherwise. Substituting $\hat{E}_t Y_{t+1}$ into the model reveals that Y_1, Y_2 is an REE.

(ii) Now consider (23). If agents observe time$-t$ information when forming time$-t$ expectations then

$$\hat{E}_t z_{t+1} = a_z + b_z z_t \tag{A35}$$

where $z \in \{\pi, x\}$. We say that (23) yields self-confirming beliefs if agents correctly understand the mean and serial correlation of x and π, i.e., $a_z = (1 - b_z)E(z_t)$, $b_z = (E(z_t z_{t-1}) - a_z E(z_t))/E(z_{t-1}^2)$. Given fixed a_z, b_z and expectations (A35), Y_t is a 2-state process: Y_j, where Y_j is the actual value of Y_t given expectations and $\epsilon_t = \epsilon_j$. This implies $E(z_t z_{t-1}) = q\bar{q}z_2^2 + ((1-q)\bar{q} + (1-p)(1-\bar{q}))z_1 z_2 + p(1-\bar{q})z_1^2$, $E(z_t^2) = \bar{q}z_2^2 + (1-\bar{q})z_1^2$, $E(z_t) = \bar{q}z_2 + (1-\bar{q})z_1$. Solving for a_z and b_z and substituting these values into (A35) yields:

$$\hat{E}_t(z_{t+1} | \epsilon_t = \epsilon_1) = pz_1 + (1-p)z_2$$
$$\hat{E}_t(z_{t+1} | \epsilon_t = \epsilon_2) = qz_2 + (1-q)z_1$$

Substituting expectations into the model and solving for z_1, z_2 straightforwardly reveals that z_1 and z_2 must be an REE. Therefore, (23) is not consistent with a non-rational equilibrium of an incoherent model if agents have current information.[37]

We conclude that if beliefs formed under PLMs (18)-(22) are self-confirming then an REE exists. Consequently, (18)-(22) are not consistent with any non-rational equilibrium of an incoherent model.

A.11   Proof of Proposition 9

To assess E-stability of each RPE, we express the RPE unconditional mean of inflation and output as a function of agents' expectations, Y^e:

$$\bar{Y}^{PP}(Y^e) = A_P Y^e + \bar{\Gamma}^{PP},$$
$$\bar{Y}^{ZP}(Y^e) = (\bar{q}A_P + (1-\bar{q})A_Z) Y^e + \bar{\Gamma}^{ZP},$$
$$\bar{Y}^{PZ}(Y^e) = ((1-\bar{q})A_P + \bar{q}A_Z) Y^e + \bar{\Gamma}^{PZ},$$
$$\bar{Y}^{ZZ}(Y^e) = A_Z Y^e + \bar{\Gamma}^{ZZ},$$

---

[37]Note that our result is related to Evans and McGough (2018a), who study E-stability of REE in linear models when agents cannot observe exogenous shocks.

where $\bar{\Gamma}^i$ collect terms that do not depend on beliefs, Y^e. It immediately follows that

$$
\begin{aligned}
DT_{\bar{Y}^{PP}} &= A_P - I, \\
DT_{\bar{Y}^{ZP}} &= \bar{q}A_P + (1 - \bar{q})A_Z - I, \\
DT_{\bar{Y}^{PZ}} &= (1 - \bar{q})A_P + \bar{q}A_Z - I, \\
DT_{\bar{Y}^{ZZ}} &= A_Z - I.
\end{aligned}
$$

It is straightforward to show that the real parts of the eigenvalues of $DT_{\bar{Y}^{PP}}$ are negative and the real part of an eigenvalue of $DT_{\bar{Y}^{ZZ}}$ is positive. Therefore, the PP RPE is always E-stable and the ZZ RPE is never E-stable.

The ZP RPE is E-stable if and only if

$$
\begin{aligned}
tr(DT_{\bar{Y}^{ZP}}) &= \beta + a - \frac{a\bar{q}\psi(\beta + a + 1)}{a\psi + 1} - 1 < 0 \\
Det(DT_{\bar{Y}^{ZP}})) &= \frac{\bar{q}a(a\psi + \psi)}{a\psi + 1} - a > 0
\end{aligned}
$$

where $tr(B)$ denotes the trace of matrix B. We have $tr(DT_{\bar{Y}^{ZP}}) < 0 < Det(DT_{\bar{Y}^{ZP}})$ if and only if $\bar{q}(1 + a)\psi - 1 - a\psi > 0$. From the proofs of Propositions 1 and 3:

$$
\begin{aligned}
\epsilon^{PP,RPE} - \epsilon_2^{ZP,RPE} &= v(\bar{q}(1 + a)\psi - 1 - a\psi)) \\
v &= \frac{a(\lambda\epsilon_2\psi + a\mu(\psi - 1))}{(1 - \bar{q})\lambda\psi(a + 1)(a(\psi - \bar{q}) + 1 - \bar{q})} > 0
\end{aligned}
$$

Therefore, if the ZP RPE is E-stable then $\epsilon^{PP,RPE} > \epsilon_2^{ZP,RPE}$ and the condition for PP existence becomes $\epsilon_1 > \epsilon^{PP,RPE}$ and the condition for ZP existence becomes $\epsilon^{PP,RPE} \geq \epsilon_1 > \epsilon_2^{ZP,RPE}$ as demonstrated in the proofs of Propositions 1 and 3.[38] Hence, if the ZP RPE exists and is E-stable then the PP solution does not exist.

Next consider the PZ solution. The PZ solution is E-stable if and only if

$$
\begin{aligned}
tr(DT_{\bar{Y}^{PZ}}) &= \frac{\beta - 2a\psi + a - 1}{a\psi + 1} + \frac{\bar{q}(\beta a\psi + a^2\psi + a\psi)}{a\psi + 1} < 0 \\
Det(DT_{\bar{Y}^{PZ}})) &= -\frac{a(1 - \psi)}{a\psi + 1} - \frac{a\bar{q}(a\psi + \psi)}{a\psi + 1} > 0
\end{aligned}
$$

which holds if and only if $0 < \psi - 1 - \bar{q}\psi(1 + a) = den^{PZ,RPE}a^{-1}$ where $den^{PZ,RPE}$ is equal to den^{PZ} defined in the Proposition 1 proof when $q = \bar{q}$ and $p = 1 - \bar{q}$. From the proof of Proposition 3, the PZ RPE only exists if $den^{PZ,RPE} < 0$. Hence the PZ RPE is never E-stable.

Therefore, the PP RPE is the only E-stable RPE solution when $\epsilon_1 > \epsilon^{PP,RPE}$, and the ZP RPE is the only E-stable RPE solution when $\epsilon^{PP,RPE} \geq \epsilon_1 > \epsilon_2^{ZP,RPE}$. It follows that a unique E-stable RPE solution exists when $\epsilon_1 > \bar{\epsilon}_{RPE}$.

---

[38]If $\bar{q} = 1$, the PP exists and is E-stable if and only if $\epsilon_1 > \epsilon^{PP,RPE}$ and the ZP exists and is E-stable if and only if $\epsilon_1 \leq \epsilon^{PP,RPE}$. The ZZ and PZ solutions cannot be E-stable.

A.12 RPE under Infinite Horizon Learning

Consider the following infinite horizon New Keynesian model:[39]

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T \geq t} \beta^{T-t} \left((1-\beta)x_{T+1} + \sigma\pi_{T+1} - \sigma\beta i_{T+1} + \epsilon_T \right) \tag{A36}$$

$$\pi_t = \lambda x_t + \hat{E}_t \sum_{T \geq t} (\xi\beta)^{T-t} \left(\xi\beta\lambda x_{T+1} + (1-\xi)\beta\pi_{T+1} \right) \tag{A37}$$

$$i_t = \max\{\psi\pi_t, -\mu\} \tag{A38}$$

where $\lambda = (1-\xi\beta)(1-\xi)/\xi$. Under infinite horizon learning, agents need to forecast the paths of the nominal interest rate and the shock, in addition to the paths of inflation and output. Consistent with the RPE studied in section 3.2, we assume that agents set endogenous and exogenous variable forecasts at all horizons equal to the unconditional means of each variable (i.e. $\hat{E}_t z_T = E(z_T)$ for all $T > t$ and $z = \pi, x, i, \epsilon$). We have:

$$E(\pi) = E\left(\lambda x_t + \hat{E}_t \sum_{T \geq t} (\xi\beta)^{T-t} \left(\xi\beta\lambda x_{T+1} + (1-\xi)\beta\pi_{T+1} \right) \right)$$

$$\implies E(x) = \frac{1-\beta}{\lambda} E(\pi)$$

and

$$\pi_t = \lambda x_t + \sum_{T \geq t} (\xi\beta)^{T-t} \left(\xi\beta\lambda E(x) + (1-\xi)\beta E(\pi) \right)$$

$$= \lambda x_t + \beta E(\pi)$$

$$\implies x_t = \lambda^{-1}(\pi_t - \beta E(\pi))$$

Substituting for x_t and also for expectations in (A36) gives an expression for RPE inflation:

$$x_t = \lambda^{-1}(\pi_t - \beta E(\pi))$$

$$= -\sigma i_t + \epsilon_t + \sum_{T \geq t} \beta^{T-t} \left((1-\beta)E(x) + \sigma E(\pi) - \sigma\beta E(i) + \beta E(\epsilon_T) \right)$$

$$\implies \pi_t = -\lambda\sigma i_t + \lambda\epsilon_t + (1 + \frac{\lambda\sigma}{1-\beta})E(\pi) - \frac{\beta\lambda\sigma}{1-\beta}E(i) + \frac{\lambda\beta}{1-\beta}E(\epsilon)$$

Let $\hat{z} = (z_1, z_2)'$ denote the vector of state-contingent RPE values of z for any variable, z. Note that $E(z) = \bar{q}z_2 + (1-\bar{q})z_1$. Then the infinite horizon RPE solution for inflation

---

[39]See Eusepi et al. (2021) for a recent derivation of the model (A36)-(A38). Note that this model collapses to the standard 3-equation model in our paper if we impose RE. Consequently, a stochastic process for inflation, output and the interest rate is an REE of (A36)-(A38) if and only if said stochastic process is an REE of (1)-(3).

satisfies:

$$\hat{\pi} = \left(1 + \frac{\lambda\sigma}{1-\beta}\right)\tilde{K}\hat{\pi} - \lambda\sigma\left(I - \beta\tilde{K}\right)^{-1}\hat{i} + \lambda\left(I - \beta\tilde{K}\right)^{-1}\hat{\epsilon}$$

where I is the identity matrix and

$$\tilde{K} = \begin{pmatrix} 1 - \bar{q} & \bar{q} \\ 1 - \bar{q} & \bar{q} \end{pmatrix}$$

Premultiplying both sides of the last equation by $\left(I - \beta\tilde{K}\right)$ and rearranging yields

$$\left(I - (1 + \lambda\sigma)\tilde{K}\right)\hat{\pi} = -\lambda\sigma\hat{i} + \lambda\hat{\epsilon} \tag{A39}$$

From the proof of Proposition 1 and 3, it can be seen that any solution of (A39) is also an RPE of (1)-(3). Hence, the infinite horizon model (A36)-(A38) admits the same RPE as (1)-(3), and therefore an incoherent model can admit RPE under infinite horizon learning under some conditions. The result is summarized in the following proposition.

Proposition 14 *Consider (A36)-(A38) and suppose $\epsilon_2 \geq 0$. Then:*

    i. An RPE exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{RPE}$, where $\bar{\epsilon}_{RPE}$ depends on the model's parameters, see Equation (A11) in the Appendix A.3, and satisfies $\bar{\epsilon}_{RPE} = -\infty$ if $q = 1$.

    ii. $\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if and only if $p + q \geq 1$.