ISSN: 2281-1346

UNIVERSITÀ DI PAVIA
Department of Economics and Management

DEM Working Paper Series

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 # 206 (06-22)

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Nonsingular M-matrices : a Tour in the Various Characterizations and in Some Related Classes

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Abstract. The main purpose of the present paper is to give a unified account of the various characterizations of nonsingular $\mathcal{M}$-matrices and a unified "path" of the mathematical proofs concerning the equivalences of the said characterizations. Some related classes of matrices are investigated and some economic applications are discussed.

Key words. $\mathcal{M}$-matrices, $\mathcal{K}$-matrices, $\mathcal{Z}$-matrices, Leontief input-output models, linear economic models.

1. Introduction

There is no need to emphasize the relevance of $\mathcal{M}$-matrices in many theoretical and applied sectors of Mathematics and Economic Analysis: indeed, this class of matrices appears in the analysis of many linear economic models, such as the classical Leontief input-output model, the various Sraffa models, but also, for example, in linear complementarity problems, finite Markov chains, theory of stochastic processes, systems of linear or nonlinear equations, etc.

It is well known that a nonsingular $\mathcal{M}$-matrix is a square (usually real) matrix C, of order n, than can be expressed as

$$C = \mu I - A,$$

where A is a nonnegative matrix (i. e. $A = [a_{ij}]$, $a_{ij} \geqq 0$, $\forall i, j = 1, ..., n$) and $\mu > \lambda^*(A)$, where $\lambda^*(A)$ is the Frobenius eigenvalue of A (i. e. its spectral radius). If, in the above representation, it holds $\mu \geqq \lambda^*(A)$, we speak of *singular* $\mathcal{M}$-matrices or also *general* $\mathcal{M}$-matrices. Of this last class of matrices only few hints will be given in the present paper.

Many authors investigated the various characterizations of nonsingular $\mathcal{M}$-matrices; we quote only the basic papers of Fiedler and Ptàk (1962), Plemmons (1977), Poole and Boullion (1974), Magnani and Meriggi (1981) and the books of Berman and Plemmons (1994), Bapat and Raghavan (1977), Seneta (1973), Windish (1989).

It seems that the term "$\mathcal{M}$-matrix" was first used by Ostrowski (1937), in honour of the German mathematician H. Minkowski. Some authors use the term "$\mathcal{K}$-matrices" (for example Fiedler and Ptàk (1962), perhaps in honour of the Russian mathematician D. M. Kotelyanskii, quoted by these authors). The main purpose of the present paper is to present a unified "path" of the mathematical proofs concerning the equivalence of the various characterizations proposed in the literature for nonsingular $\mathcal{M}$-matrices.

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Indeed, apart from the complete proof given by Fiedler and Ptàk (1962), who, however, take into consideration a relatively small group of characterizations, the other authors give partial and synthetic proofs, which often make reference to previous results. For example, Bapat and Raghavan (1997) use properties and results of the theory of matrix games; Magnani and Meriggi (1981) give a quite long list of characterizations, with several interesting comments, but with no proofs.

The paper is organized as follows. In Section 2 we precise the notations and the main definitions used in the sequel of the paper. Section 3 contains the characterizations of nonsingular $\mathcal{M}$-matrices taken into consideration in the present paper. Section 4 is concerned with the proofs of the equivalences of the various characterizations. In Section 5 some extensions of the class of nonsingular $\mathcal{M}$-matrices are discussed, together with some connections of $\mathcal{M}$-matrices with other classes of matrices. In the final Section 6 some economic applications are discussed.

2. Notations and Main Definition

Unless otherwise stated, all matrices such as $A = [a_{ij}]$, are real square matrices of order n, and vectors such as x, are real vectors of n elements, i. e. $x \in \mathbb{R}^n$. Vectors are considered as column vectors, so the transpose of x, denoted by $x^\top$, is a row vector. The transpose of A is denoted by $A^\top$; x^* is the conjugate transpose of x ($x^* = x^\top$ if x is real). A_i, $i = 1, ..., n$, denotes the i-th *row* of A, wheeras A^j, $j = 1, ..., n$, denotes the j-th column of A. By $[0]$ we denote the matrix or the vector whose all entries are zero. By $e \in \mathbb{R}^n$ we denote the *sum vector,* i. e. the vector whose all entries are 1 :

$$e^\top = [1, 1, , ..., 1] .$$

Given two matrices A, B of the same order (not necessarily square), we put

- $A \geqq B$, if $a_{ij} \geqq b_{ij}$, $\forall i, j$.
- $A \geq B$, if $A \geqq B$, $A \neq B$.
- $A > B$, if $a_{ij} > b_{ij}$, $\forall i, j$.

In particular, if $B = [0]$, the above inequalities characterize, respectively, a *nonnegative matrix,* a *semipositive matrix* and a *positive matrix.*

The same conventions are used to compare two vector of $\mathbb{R}^n$.

The notations $\ngeqq, \ngeq, \ngtr$, are used to denote the reverse properties. The notations $\leqq, \leq, <$ $, \nleqq, \nleq, \nless$ are now clear.

If A is a square real matrix of order n, then its *main diagonal* is described by $a_{11}, a_{22}, ..., a_{nn}$, whereas the elements a_{ij}, $i \neq j$, are said *extra-diagonal elements* or also *off-diagonal elements.* If it holds

$$i < j \Longrightarrow a_{ij} = 0,$$

then A is an *upper triangular matrix,* whereas if

$$i > j \Longrightarrow a_{ij} = 0,$$

then A is a *lower triangular matrix.*

The class of $\mathcal{Z}$*-matrices* is the class of square matrices with nonpositive extra-diagonal elements:

$$A \in \mathcal{Z} \iff \{i \neq j \Longrightarrow a_{ij} \leqq 0\}\,.$$

We denote by $\mathcal{Z}^+$ the class of $\mathcal{Z}$-matrices with a positive (main) diagonal:

$$A \in \mathcal{Z}^+ \iff \{A \in \mathcal{Z},\ a_{ii} > 0,\ \forall i = 1, ..., n\}\,.$$

By $\mathcal{D}$ we denote the class of (square) *diagonal matrices:*

$$A \in \mathcal{D} \iff \{i \neq j \Longrightarrow a_{ij} = 0\}\,.$$

$\mathcal{D}^+ \subset \mathcal{D}$ is the proper subset of $\mathcal{D}$ with a positive diagonal:

$$A \in \mathcal{D}^+ \iff \{A \in \mathcal{D},\ a_{ii} > 0,\ \forall i = 1, ..., n\}\,.$$

Obviously, the identity matrix I is contained in $\mathcal{D}^+$.

By $diag(a_i)$, $i = 1, ..., n$, we intend the diagonal matrix with its main diagonal formed by $a_1,\ a_2, ..., a_n$.

P denotes a *permutation matrix,* i. e. a matrix obtained by permuting the rows (or the columns) of the identity matrix I. Note that P is an *orthogonal matrix,* i. e. $P^{-1} = P^\top$.

Given the square matrix A, of order n, we call

• *principal minor of order k of A* every determinant of order k obtained from A by considering k rows of A and the *corresponding* k columns ($k = 1, ..., n$). We use the notations:

$$D_1, D_2, ..., D_k, ..., D_n = \det(A).$$

We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

principal minors of order k. As $\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$, we have on the whole $2^n - 1$ principal minors from A of order n. Following Gale and Nikaido (1965) and Fiedler and Ptàk (1966b), a square matrix A of order n, is called a $\mathcal{P}$-matrix or it belongs to the $\mathcal{P}$-class, if all its principal minors are positive.

• *leading principal minor of order k* (or *North-West principal minor of order k* or *successive principal minor of order k*) of A, that determinant of order k obtained from A by considering the *first* k rows and the *first* k columns of A ($k = 1, ..., n$). We use the notations:

$$\Delta_1 = a_{11},\ \Delta_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \Delta_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, ..., \Delta_n = \det(A).$$

We say that the real square matrix A is *quasi-positive definite* (*quasi-negative definite*) if $x \neq [0] \Longrightarrow x^\top Ax > 0$ ($x^\top Ax < 0$); if A is *symmetric,* the previous inequalities characterize, respectively, the class of *positive definite matrices* and the class of *negative definite matrices.*

Note that A is quasi-positive definite (resp. quasi-negative definite) if and only if $(A+A^\top)$ is positive definite (resp. negative definite), i. e. on the grounds of well-known criteria, if and only if every eigenvalue of $(A + A^\top)$ is positive (negative) or if and only if the sequence of all n leading principal minors of $(A + A^\top)$ is formed by positive elements (by elements alternating in sign, with the first element negative).

Let be given $A \geqq [0]$, A square of order n; we denote by $\lambda^*(A)$ its *Frobenius eigenvalue* (or *Frobenius root*), i. e. $\lambda^*(A) = \max_{\lambda \in \sigma(A)} |\lambda|$, being $\sigma(A)$ the *spectrum of* A :

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I) = 0\} .$$

It is well known that $\lambda^*(A)$ is a real root of the characteristic equation of $A \geqq [0]$.

The square matrix A (not necessarily nonnegative) is a *small matrix* or a *convergent matrix* if its *spectral radius*

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

is less than unity: $\rho(A) < 1$. In this case we have (see, e. g., Varga (1962))

$$\lim_{k \longrightarrow +\infty} (A)^k = [0] ,$$

where $(A)^k$ denotes the power of A of order k.

We recall that a square matrix A, of order n, is said to be *decomposable* or *reducible* if, for some proper nonempty subset J of $N = \{1, 2, ..., n\}$ it holds $a_{ij} = 0$ for $i \in J$ and $j \notin J$. A is *indecomposable* or *irreducible* if A is not decomposable. Equivalently: A is decomposable if and only if there exists a permutation matrix P such that

$$PAP^\top = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ,$$

with A_{11} square submatrix of A (and hence also A_{22} square) and at least one of the submatrices A_{12}, A_{21} a zero matrix.

A powerful generalization of the previous characterization is given by the *Gantmacher normal form of* A (see Gantmacher (1959)). We have the said normal form when, by means of a suitable permutation matrix P, which always exists, the matrix A is put into the form

$$PAP^{\top} = \begin{bmatrix} A_{11} & [0] & \cdots & [0] & [0] & [0] & \cdots & [0] \\ [0] & A_{22} & \cdots & [0] & [0] & [0] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ [0] & [0] & \cdots & A_{gg} & [0] & [0] & \cdots & [0] \\ A_{g+1,1} & A_{g+1,2} & \cdots & A_{g+1,g} & A_{g+1,g+1} & [0] & \cdots & [0] \\ A_{g+2,1} & A_{g+2,2} & \cdots & A_{g+2,g} & A_{g+2,g+1} & A_{g+2,g+2} & \cdots & [0] \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{s1} & A_{s2} & \cdots & \cdots & \cdots & \cdots & \cdots & A_{ss} \end{bmatrix}$$

where:

i) Each submatrix $A_{11}, A_{22}, ..., A_{ss}$ ("principal blocks") is square and indecomposable (if a submatrix is of order 1 it may be the zero element).

ii) If A is indecomposable, then $PAP^{\top} = IAI = A = A_{11}$, i. e. $s = g = 1$.

iii) If $s = g > 1$, then

$$PAP^{\top} = \begin{bmatrix} A_{11} & [0] & \cdots & [0] \\ [0] & A_{22} & \cdots & [0] \\ \vdots & \vdots & \cdots & \vdots \\ [0] & [0] & \cdots & A_{ss} \end{bmatrix},$$

i. e. we have a so-called *completely decomposable matrix* or *diagonal-blocks matrix.*

iv) With $s > g$ it holds

$$B_t = [A_{t1}, A_{t2}, ..., A_{t,t-1}] \neq [0], \ \forall t = g+1, ..., s.$$

The Gantmacher normal form is unique, up to permutations within the principal blocks. See Gantmacher (1959).

Another important concept related to square matrices (also complex) and to $\mathcal{M}$-matrices, is given by matrices with *dominant diagonals.* The original definition has been developed mainly by Hadamard (1903).

Definition 1. A square real matrix $A = [a_{ij}]$ of order n is said to possess a *row dominant diagonal* if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad \forall i = 1, ..., n;$$

A has a *column dominant diagonal* if

$$|a_{jj}| > \sum_{i \neq j} |a_{ij}|, \ \forall j = 1, ..., n.$$

If, moreover, $a_{ii} > 0$ ($a_{ii} < 0$), then A is said to possess, respectively, a positive (a negative) row dominant diagonal or a positive (negative) column dominant diagonal.

The next definition is due to McKenzie (1960), even if also quite recently it has been rediscovered by researchers in Linear Algebra and Matrix Theory.

Definition 2. A square real matrix A of order n has a *row quasi-dominant diagonal* (resp. a *column quasi-dominant diagonal*) if there exists a diagonal matrix $D \in \mathcal{D}^+$ such that AD (or DA) has a row dominant diagonal (or a column dominant diagonal) in the sense of Hadamard, i. e. of Definition 1. In other words, there exist numbers

$$d_1 > 0,\ d_2 > 0, ..., d_n > 0,$$

such that it holds, respectively,

$$d_i \left| a_{ii} \right| > \sum_{j \neq i} d_j \left| a_{ij} \right|, \quad \forall i = 1, ..., n; \tag{1}$$

$$d_j \left| a_{jj} \right| > \sum_{i \neq j} d_i \left| a_{ij} \right|, \quad \forall j = 1, ..., n. \tag{2}$$

We have to note that (unlike Definition 1) if A has a row quasi-dominant diagonal, then it has a column quasi-dominant diagonal and vice-versa (see, e. g., Giorgi and Zuccotti (2009), Kemp and Kimura (1978), Magnani (1972-73)). Therefore it is convenient to speak only of "matrices with a quasi-dominant diagonal". If, in addition, $a_{ii} > 0$, $i = 1, ..., n$, then A is said to possess a *positive quasi-dominant diagonal.* Similarly, if, in addition, $a_{ii} < 0$, $i = 1, ..., n$, then A is said to possess a *negative quasi-dominant diagonal.* The following properties are well-known.

1) Matrices with a dominant diagonal are a proper subclass of matrices with a quasi-dominant diagonal.

2) A matrix with a quasi-dominant diagonal is nonsingular.

3) If A (real and square) has a positive quasi-dominant diagonal, then all its principal minors are positive (see also Ostrowski (1937)).

4) If A is *indecomposable,* then relations (1) and (2) can be rewritten with *weak inequalities, but with at least one strict inequality.*

We recall also some notions on the "splits" of a square matrix A, of order n. First we note that this matrix can be always expressed in a unique way, in the form

$$A = D(A) + L(A) + U(A) \tag{3}$$

where $D(A) = diag(a_{ii})$, $L(A)$ is a lower triangular matrix

$$L(A) = [l_{ij}] = \begin{cases} 0, & \text{if } j \geqq i \\ a_{ij}, & \text{if } j < i, \end{cases}$$

and $U(A)$ is an upper triangular matrix

$$U(A) = [u_{ij}] = \begin{cases} 0, & \text{if } j \leqq i \\ a_{ij}, & \text{if } j > i. \end{cases}$$

Every expression of the type (C, B and A square matrices of the same order)

$$C = B - A$$

is called a "split" of the matrix C. Obviously, a $\mathcal{Z}$-matrix C can be always expressed as

$$C = D - A,$$

with $D \in \mathcal{D}$, $A \geqq [0]$, hence in the representation (3) of a $\mathcal{Z}$-matrix C it holds

$$L(C) \leqq [0]\,,\ U(C) \leqq [0]\,.$$

Following Varga (1976), we put, with A square of order n,

$$J_\alpha(A) = \alpha\,[D(A)]^{-1}\,(L(A) + U(A)) + (1-\alpha)I, \quad \alpha > 0,$$

$$T_\alpha(A) = (D(A) - \alpha L(A))^{-1}\,\{(1-\alpha)D(A) + \alpha U(A)\}\,, \quad \alpha > 0,$$

$$V_\alpha(A) = (D(A) - \alpha U(A))^{-1}\,\{(1-\alpha)D(A) + \alpha L(A)\}\,(D(A) - \alpha L(A))^{-1} \cdot$$
$$\cdot\,\{(1-\alpha)D(A) + \alpha U(A)\}\,, \quad \alpha > 0.$$

$J_\alpha(A)$ is said "point Jacobi overrelaxation iteration matrix"; $T_\alpha(A)$ is said "point-successive overrelaxation iteration matrix" and $V_\alpha(A)$ is said "point-symmetric successive overrelaxation matrix".

Always with A square, of order n, we denote by $|A|$ the matrix whose all entries are the moduli of the entries of A :

$$|A| = [\alpha_{ij}] = |a_{ij}|\,, \quad i, j = 1, ..., n.$$

In order to make no confusion with the determinant of A, we shall denote this last one by $\det(A)$.

$\Omega(A)$ is the *equimodular set* for A : if B is square of order n, then $B \in \Omega(A)$ if and only if $|B| = |A|$.

Z_A is the *comparison matrix* of A, i. e.

$$Z_A = \begin{cases} |a_{ii}|\,, & \text{if } i = j \\ -|a_{ij}|\,, & \text{if } i \neq j. \end{cases}$$

Obviously $Z_A \in \mathcal{Z}$ and by this convention, A has a row dominant diagonal (a column dominant diagonal) if and only if $Z_A e > [0]$ (if and only if $e^\top Z_A > [0]$), whereas A has a row quasi-dominant diagonal (in the sense of McKenzie) if and only if the system

$$\begin{cases} Z_A x > [0] \\ x > [0] \end{cases}$$

has a solution. A will have a column quasi-dominant diagonal if and only if the system

$$\begin{cases} y^\top Z_A > [0] \\ y > [0] \end{cases}$$

has a solution.

The matrix A, of order (m, n), belongs to the *class* $\mathcal{S}$ or is an $\mathcal{S}$-matrix if $Ax > [0]$ for some $x > [0]$. Equivalently: for some $x \geq [0]$ or for some $x \geqq [0]$ I.ndeed, if x^0 is a nonnegative or a semipositive vector such that $Ax^0 > [0]$, we can consider the vector $\bar{x} = x^0 + \alpha e$, $\alpha \geqq 0$. Then we have $\bar{x} > [0]$, $\forall \alpha > 0$ and, moreover, there exist values of $\alpha \geqq 0$ such that $y = A\bar{x} > [0]$, being y a continuous function of α.

Other definitions and conventions will be recalled in the next sections, when needed.

3. Characterizations of Nonsingular M-matrices

In the present section we give a survey of the main characterizations of nonsingular $\mathcal{M}$-matrices. We try to operate distinctions among the various "groups" which have one or more properties in common. We recall that we consider a real n-square $\mathcal{Z}$-matrix $C = [c_{ij}]$, with $c_{ij} \leqq 0$, $\forall i \neq j$; each item of the following list of characterizations is equivalent to the proposition: "the $\mathcal{Z}$-matrix C is a nonsingular $\mathcal{M}$-matrix".

1. First Group. This group takes into consideration linear inequality or equality systems, with sign conditions.

$(C1)$ The system $Cx > [0]$ admits a solution $x \geqq [0]$.

$(C2)$ The system $Cx > [0]$ admits a solution $x > [0]$.

$(C3)$ There exists $y > [0]$ such that $Cx = y$ admits a solution $x \geqq [0]$.

$(C4)$ For any $y \geqq [0]$ the system $Cx = y$ admits a solution $x \geqq [0]$.

Nikaido (1968, 1970) calls $(C1)$ (and the equivalent condition $(C3)$) the *weak solvability condition,* with reference to a Leontief linear economic model, and calls $(C4)$ the *strong solvability condition.*

$(C5)$ There exists a matrix $D \in \mathcal{D}^+$ such that $CDe > [0]$.

$(C6)$ C is a *generalized positive matrix* in the sense of Varga (1976a), i. e. there exists a vector $x > [0]$ such that $Cx \geq [0]$ and if $C_i x = 0$, then C admits a chain connecting i with some j, such that $C_j x > 0$ (we say that C admits the said chain if $N = \{1, 2, ..., n\}$ contains indices $i = i_1, i_2, ..., i_h = j$ such that $c_{i_k i_{k+1}} \neq 0$, $\forall k = \{1, 2, ..., h-1\}$).

$(C7)$ There exists a vector $x > [0]$ such that

$$\begin{cases} Cx \geqq [0] \\ [C - U(C)]\, x > [0]\,, \end{cases}$$

i. e. such that

$$\begin{cases} Cx \geqq [0] \\ [D(C) + L(C)]\, x > [0]\,. \end{cases}$$

$(C8)$ The system

$$\begin{cases} Cx \leqq [0] \\ x \geq [0]\,, \end{cases}$$

admits no solution x.

Second Group. This group is concerned with properties of determinants of matrices associated to C, in particular principal minors and leading principal minors.

$(C9)$ All leading principal minors of C are positive:

$$\Delta_1 > 0,\ \Delta_2 > 0, ..., \Delta_n > 0.$$

$(C10)$ All principal minors of order k, D_k, $k = 1, ..., n$, are positive.

In economic analysis conditions $(C9)$ and $(C10)$ are known as "Hawkins-Simon conditions"(Hawkins and Simon (1949)), developed within the researches on the celebrated "Leontief input-output model". Hawkins and Simon (1949) proved condition $(C10)$ as an equivalent condition for C to be a nonsingular $\mathcal{M}$-matrix; however, this condition had already been considered by Ostrowski (1937). Condition $(C9)$ was proved by Geogescu-Roegen (1951, 1966) who proved also the equivalence between $(C9)$ and $(C10)$. Following Gantmacher (1959), this equivalence has also been proved by Kotelianskii (1952). Good proofs concerning conditions $(C9)$ and $(C10)$ are given by Nikaido (1968, 1970), Kemp and Kimura (1978), Takayama (1985), Woods (1978). See also the important paper of Debreu and Herstein (1953).

$(C11)$ Let $N_1, N_2, ..., N_n$ be nonempty subsets of $N = \{1, 2, ..., n\}$, with $N_1 \subset N_2 \subset ... \subset N_n = N$, and $M_{(i)} = [c_{ij}]\,,\, i, j \in N_i$. Then $\det(M_{(i)}) > 0$, $\forall i \in N$.

$(C12)$ The matrix $F = -C$ verifies the *Routh-Hurwitz stability criterion,* i. e. if we denote by s_i the sum of all $\begin{pmatrix} n \\ i \end{pmatrix}$ principal minors of order i of F, and we put

$$k_i = \begin{cases} (-1)^i s_i,\ \forall i \in N = \{1, 2, ..., n\} \\ 0,\ \forall i > N, \end{cases}$$

it holds

$$\det\begin{pmatrix} k_1 & k_3 & k_5 & \cdots & k_{2i-1} \\ 1 & k_2 & k_4 & \cdots & k_{2i-2} \\ 0 & k_1 & k_3 & \cdots & k_{2i-3} \\ 0 & 1 & k_2 & \cdots & k_{2i-4} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & k_i \end{pmatrix} > 0, \quad \forall i \in N.$$

Third Group. The characterizations of this group are concerned with the possibility of representing the matrix C by means of a "split", i. e. as a difference of two matrices, of the type

$$C = B - A.$$

$(C13)$ There exists a matrix $A \geqq [0]$ and a real scalar μ such that

$$C = \mu I - A, \quad \mu > \lambda^*(A).$$

We remark that it is always possible to write the $\mathcal{Z}$-matrix C in the above form, i. e. $C = \mu I - A$, $\mu \in \mathbb{R}$, $A \geqq [0]$. Indeed, it is sufficient to choose

$$\mu \geqq \max_i \{c_{ii}\},$$

and then to choose the matrix

$$A = \mu I - C$$

which is, by construction, a nonnegative matrix.

$(C14)$ There exists a real number μ such that

$$(\mu I - C) \geqq [0], \ \mu > \lambda^*(\mu I - C).$$

$(C15)$ There exists a real number $\mu \geqq \max_{i \in N} \{c_{ii}\}$ such that $\mu > \lambda^*(\mu I - C)$.

$(C16)$ It holds the implication

$$C = (\mu I - A), \quad A \geqq [0] \Longrightarrow \mu > \lambda^*(A).$$

$(C17)$ It holds the implication

$$\mu \geqq \max_{i \in N} \{c_{ii}\} \Longrightarrow \mu > \lambda^*(\mu I - C).$$

$(C18)$ If $R \geqq C$, with $R \in \mathcal{D}$, then R^{-1} exists and, with $D = diag(c_{ii})$, the matrix $R^{-1}(D-C)$ is a small matrix.

$(C19)$ There exists a real number μ and a matrix A such that $\mu > 0$, $A \geqq [0]$, $C = [\mu I - A]$, and the series

$$\frac{1}{\mu}\sum_{k=0}^{+\infty}\left(\frac{1}{\mu}A\right)^k$$

is convergent (this series converges to $(\mu I - A)^{-1} = C^{-1} \geq [0]$; see $(C31)$).

$(C20)$ It holds the following implication:

$$\{\mu > 0,\ A \geqq [0]\,,\ C = \mu I - A\} \Longrightarrow \frac{1}{\mu}\sum_{k=0}^{+\infty}\left(\frac{1}{\mu}A\right)^{k} \text{ is convergent.}$$

$(C21)$ The matrix C admits the split

$$C = B - A,$$

with A and/or B is in the class of $\mathcal{S}$-matrices, B^{-1} exists and $B^{-1}A$ is a nonnegative small matrix.

4. Fourth Group. This group is concerned with characterizations where the matrix C is decomposed into the product of two matrices.

$(C22)$ There exist a permutation matrix P and two $\mathcal{Z}$-matrices R and S, R lower triangular and S upper triangular, both with a positive diagonal, such that

$$PCP^{\top} = RS.$$

$(C23)$ There exist two $\mathcal{Z}$-matrices R and S, R lower triangular and S upper triangular, both with a positive diagonal and with all leading principal minors positive, such that

$$C = RS.$$

5. Fifth Group. This group is concerned with characterizations of nonsingular $\mathcal{M}$-matrices which involve properties of quadratic forms.

$(C24)$ There exists a positive definite (symmetric) matrix A such that AC is quasi-positive definite.

$(C25)$ There exists a positive (symmetric) matrix A such that the symmetric matrix

$$(AC + C^{\top}A)$$

has all its eigenvalues positive.

$(C26)$ There exists a positive definite (symmetric) matrix A such that

$$(AC + C^{\top}A)$$

has all its leading principal minors positive.

$(C27)$ There exist two matrices $D \in \mathcal{D}^{+}$ and $E \in \mathcal{D}^{+}$ such that

$$DCE$$

is quasi-positive definite.

($C28$) There exists a matrix $D \in \mathcal{D}^+$ such that DC is quasi-positive definite.

($C29$) There exists a positive definite (symmetric) matrix A such that

$$(AC + C^\top A)$$

is a $\mathcal{Z}$-matrix and an $\mathcal{S}$-matrix.

($C30$) There exists a positive definite (symmetric) matrix $A \geqq [0]$ such that

$$(AC + C^\top A)$$

is a $\mathcal{Z}$-matrix and an $\mathcal{S}$-matrix.

6. Sixth Group. The characterizations of this group are concerned with properties of the inverse of C or of matrices related to the said inverse.

($C31$) The inverse of C exists and it holds $C^{-1} \geq [0]$.

($C32$) If $W \in \mathcal{Z}$, $W \geqq C$, then W^{-1} exists.

($C33$) If $W \in \mathcal{Z}^+$, $W \geqq C$, then W^{-1} exists and $C \in \mathcal{Z}^+$.

($C34$) There exist two nonnegative matrices R and S such that $(RCS)^{-1}$ exists and it holds $(RCS)^{-1} \geq [0]$.

7. Seventh Group. This group is concerned with characterizations related to the spectrum of C or to a particular transformation of C.

($C35$) Each real eigenvalue of C is positive:

$$\{\det(C - \lambda I) = 0,\ \lambda \in \mathbb{R}\} \Longrightarrow \lambda > 0.$$

($C36$) each eigenvalue of C has a positive real part:

$$\det(C - \lambda I) = 0 \Longrightarrow \mathrm{Re}(\lambda) > 0.$$

In other words, if $F = (-C)$, then F is a *stable matrix.* Some authors call condition (C36), the condition characterizing *positive stable matrices.* Again: if $C \in \mathcal{Z}$, then $F = (-C)$ is called in Economic Analysis a *Metzler matrix* or *Metzlerian matix,* in honour of the American economist L. A. Metzler; see Metzler (1945).

($C37$) The matrix $D = diag(c_{ii}) \in \mathcal{D}^+$ and $[I - D^{-1}C]$ is a small nonnegative matrix.

($C38$) Let A be a (symmetric) positive definite matrix. Then each eigenvalue of the *Hadamard product* $A * C \equiv \{a_{ij}c_{ij}\}$ has a positive real part.

8. Eighth Group. This group of characterizations makes reference to properties of the main diagonal of the matrix C.

($C39$) The matrix C has a positive and quasi-dominant diagonal (in the sense of McKenzie).

($C40$) There exists $D \in \mathcal{D}^+$ such that CD has a positive row dominant diagonal.

($C41$) To every real vector $x \neq [0]$ it corresponds a diagonal matrix $D = D(x) \in \mathcal{D}^+$ such that $x^\top DCx > 0$.

($C42$) To every real vector $x \neq [0]$ it corresponds a diagonal matrix $D = D(x)$, with a nonnegative diagonal, such that $x^\top DCx > 0$.

($C43$) To every complex vector $x \neq [0]$ it corresponds a diagonal matrix $D = D(x) \in \mathcal{D}^+$ such that $\mathrm{Re}(x^* DCx) > 0$.

9. Ninth Group. This group is concerned with characterizations of nonsingular $\mathcal{M}$-matrices, expressed in terms of particular implications.

($C44$) For every vector $x \neq [0]$ there exists always an index k such that $x_k(C_k x) > 0$.

($C45$) The following implication holds:

$$x \geq [0] \Longrightarrow Cx \nleqq [0] \, .$$

($C46$) The matrix C "reverses the sign" of the zero vector only, i. e. it holds:

$$x_i(C_i x) \leqq 0, \; \forall i \in N = \{1, 2, ..., n\} \Longrightarrow x = [0] \, .$$

($C47$) It holds the implixcation

$$Cx \leqq [0] \Longrightarrow x \ngeq [0] \, .$$

($C48$) It holds the implication

$$Cx \geqq [0] \Longrightarrow x \nleq [0] \, .$$

($C49$) It holds the implication

$$Cx \geqq [0] \Longrightarrow x \geqq [0] \, , \text{ for all } x \in \mathbb{R}^n .$$

This property is usually described by saying that C is a *monotone matrix.* See Collatz (1952), Mangasarian (1968).

($C50$) The following implication holds:

$$D \in \mathcal{D} \Longrightarrow \left\{ (DC)^{-1} \geq [0] \iff D \in \mathcal{D}^+ \right\} .$$

10. Tenth Group. The characterizations of this group are due to Varga (1976a) and are all referred to the use of the so-called "over-relaxation iteration matrices", important in the iterative solution of systems of linear equations of high dimensions. See, e. g., Koehler, Whinston and Wright (1975).

($C52$) For any $A \in \Omega(C)$ it holds $\rho(J_1(A)) \leqq \rho(|J_1(A)|) = \rho(J_1(Z_C)) < 1$.

$(C53)$ For any $A \in \Omega(C)$ and any $0 < \alpha < 2 \diagup [1 + \rho(J_1(A))]$ it holds

$$\rho(J_\alpha(A)) \leqq \alpha\rho(J_1(A)) + |1 - \alpha| < 1.$$

$(C54)$ For any $A \in \Omega(C)$ and any $0 < \alpha < 2 \diagup [1 + \rho(|J_1(A)|)]$ it holds

$$\rho(T_\alpha(A)) \leqq \alpha\rho(|J_1(A)|) + |1 - \alpha| < 1.$$

$(C55)$ For any $A \in \Omega(C)$ and any $0 < \alpha < 2 \diagup [1 + \rho(|J_1(A)|)]$ it holds $\rho(V_\alpha(A)) < 1$.

The previous list, taken essentially from Magnani and Meriggi (1981), has not the pretention to be complete. There are some other characterizations of nonsingular $\mathcal{M}$-matrices, due, for example, to Poole and Boullion (1974), Plemmons (1977), Berman and Plemmons (1994). Some of them are:

$(C56)$ For each *signature matrix* S there exists $\bar{x} > [0]$ such that

$$SCS > [0]\,.$$

(A signature matrix S is a square matrix $S \in \mathcal{D}$ with diagonal entries ± 1).

$(C57)$ The matrix C is nonsingular and $C + D$ is nonsingular for each $D \in \mathcal{D}^+$.

$(C58)$ The matrix $C + \alpha I$ is nonsingular for each $\alpha \geqq 0$.

$(C59)$ The matrix $C + I$ is nonsingular and

$$G = (C + I)^{-1}(C - I)$$

is convergent.

$(C60)$ The inequalities $Cx \leqq [0]\,,\, x \geqq [0]$ have only the trivial solution and C is nonsingular.

$(C60)$ has been used by Bergthaller and Dragomirescu (1971) and by Giorgi (1987) to prove the workability of the classical Leontief system. Note that if $Cx \leqq [0]$ has only the trivial solution $x = [0]\,$, C must be nonsingular. Bergthaller and Dragomirescu use the equivalent condition:

"the system $Cx \leqq [0]\,, x \geq [0]$ has no solution",

which is characterization $(C8)$ (and also $(C47)$).

We note that a necessary condition for $C \in \mathcal{Z}$ to be a nonsingular $\mathcal{M}$-matrix is that $diag(c_{ii}) > 0$, for all $i = 1, ..., n$. We remark also that a matrix $C \in \mathcal{Z}$ is a nonsingular $\mathcal{M}$-matrix if and only if each principal submatrix of C is a nonsingular $\mathcal{M}$-matrix and thus satisfies one of the equivalent conditions listed above.

Rather important is the following result, pointed out by Magnani and Meriggi (1981).

Theorem 1. The classes of $\mathcal{Z}$-matrices and nonsingular $\mathcal{M}$-matrices are algebraically closed under the following transformations:

i) $f_1(C) = C^\top$;

ii) $f_2(C) = DCE$, with $D \in \mathcal{D}^+$, $E \in \mathcal{D}^+$;

iii) $f_3(C) = PCP^\top$, with P any permutation matrix.

In other words, we have the following further characterizations of nonsingular $\mathcal{M}$-matrices:

($C61$) $C^\top$ is a nonsingular $\mathcal{M}$-matrix;

($C62$) For any $D, E \in \mathcal{D}^+$, the matrix DCE is a nonsingular $\mathcal{M}$-matrix;

($C63$) For any permutation matrix P, the matrix $PCP^\top$ is a nonsingular $\mathcal{M}$-matrix.

Proof. The fact that the class of $\mathcal{Z}$-matrices is closed under the above transformations is trivial: it is immediate to note that these transformations preserve both the sign and the diagonal and extra-diagonal position of every element c_{ij} of C. In order to prove that the same properties hold for the class of nonsingular $\mathcal{M}$-matrices, it is convenient to make reference to characterization ($C31$) : the inverse of C exists and it holds $C^{-1} \geq [0]$ (and hence C^{-1} has all its lines semipositive). The determinant of $f(C)$ becomes, respectively,

$$\det(f_1(C)) = \det(C^\top) = \det(C);$$

$$\det(f_2(C)) = \det(DCE) = \det(D)\det(C)\det(E);$$
$$\det(f_3(C)) = \det(PCP^\top) = \det(P)\det(C)\det(P^\top) = \det(C).$$

Hence $f(C)$ is nonsingular if and only if C is nonsingular. The inverse of $f(C)$ becomes, respectively,

$$[f_1(C)]^{-1} = (C^\top)^{-1} = (C^{-1})^\top;$$
$$[f_2(C)]^{-1} = (DCE)^{-1} = E^{-1}C^{-1}D^{-1};$$
$$[f_3(C)]^{-1} = (PCP^\top)^{-1} = (P^\top)^{-1}C^{-1}P^{-1} = PC^{-1}P^\top.$$

In the first case the lines of the inverse matrix are simply transposed, but their sign does not vary. In the second case the element $[c_{ij}]^{-1}$ of C^{-1} has the same position in $[f_2(C)]^{-1}$, but here the element is divided by $d_{ii}e_{jj}$, with $d_{ii} \in D$ and $e_{jj} \in E$. As $D, E \in \mathcal{D}^+$, the signs are conserved. In the third case the permutation which generates $f_3(C)$ from C is the same for $[f_3(C)]^{-1}$, which has therefore the same signs of C^{-1}. Note that this allows to rewrite ($C9$) as ($C10$). . □

The transformation $f_1(C)$ allows to rewrite certain characterizations in their "dual" form: for example, characterizations ($C1$), ($C2$), ($C3$), and ($C4$) which make reference to the "productivity" of a linear economic model, such as the classical Leontief model, can be rewritten in terms of "profitability" for the same model. In other words, for a linear economic model of the Leontief type, i. e. without joint production, productivity and profitability are equivalent properties. We have:

($C64$) There exists a vector p which solves the system

$$\begin{cases} p^\top C > [0] \\ p \geqq [0] \, . \end{cases}$$

($C65$) There exists a vector p which solves the system

$$\begin{cases} p^\top C > [0] \\ p > [0] . \end{cases}$$

($C66$) There exists a vector $v > [0]$ such that the system

$$\begin{cases} p^\top C = v^\top \\ p \geqq [0] \end{cases}$$

has a solution p.

($C67$) For any vector $v \geqq [0]$ the system

$$\begin{cases} p^\top C = v^\top \\ p \geqq [0] \end{cases}$$

has a solution p.

The transformation $f_2(C)$ is useful to obtain other characterizations of nonsingular $\mathcal{M}$-matrices, with the choice $D \in \mathcal{D}^+$, $E = I$. We recall that in economic analysis a real square matrix $F = [f_{ij}]$ is called a *Metzler matrix* or *Metzlerian matrix* if $f_{ij} \geqq 0$, $\forall i \neq j$, i. e. if $-F \in \mathcal{Z}$. It is well known (see, e. g., Kemp and Kimura (1978)) that if F is Metzlerian, then it is stable if and only if it is *D-stable,* i. e. DF is stable, $\forall D \in \mathcal{D}^+$. In economic terms this means that a Metzlerian equilibrium is stable regardless of the choice of the "adjustmeent speeds". See, e. g., Quirk and Saposnik (1968). These last authors call "totally stable" a square matrix with every principal submatrix which is D-stable. In the Metzlerian case, also total stability is equivalent to stability. Hence, also on the grounds of the transformation $f_2(C)$ and on what said on the principal submatrices of nonsingular $\mathcal{M}$-matrices, we have the following further two characterizations of nonsingular $\mathcal{M}$-matrices.

($C68$) The matrix $F = (-C)$ is D-stable.

($C69$) The matrix $F = (-C)$ is totally stable.

For a survey on stable and D-stable matrices in economic theory the reader is referred to Giorgi (2003) and to Giorgi and Zuccotti (2015a).

The transformation $f_3(C)$ is useful to justify what previously said on the principal submatrices of a nonsingular $\mathcal{M}$-matrix. In particular, it is possible to obtain the following characterization of a nonsingular $\mathcal{M}$-matrix.

($C70$) Every "principal block" of the *Gantmacher normal form* of C is a nonsingular $\mathcal{M}$-matrix.

If $C \in \mathcal{Z}$ is an *indecomposable matrix,* some of the previous characterizations can be reformulated in a slightly different form. For example:

- In characterizations ($C1$) and ($C2$), instead of $Cx > [0]$ it is possible to impose $Cx \geq [0]$.
- In characterization ($C3$), instead of $y > [0]$ it is possible to impose $y \geq [0]$.

• In characterization $(C31)$ it is possible to impose $C^{-1} > [0]$.

Hence, if $C = (\mu I - A)$, with $A \geqq [0]$ and indecomposable, it will be $(\mu I - A)^{-1} > [0]$ if and only if $\mu > \lambda^*(A)$. This result is often presented as a corollary of the celebrated Perron-Frobenius theorem for nonnegative indecomposable matrices. See, e. g., the classical paper of Debreu and Herstein (1953). These authors give also a short proof of the equivalence:

$$(\mu I - A)^{-1} > [0] \iff \{(\mu I - A) \text{ has all its principal minors positive}\}.$$

Moreover, Berman and Plemmons (1994) prove the following result, which may be considered a "reverse" statement of what previously remarked.

Theorem 2. Let $A \geqq [0]$ be a square matrix of order n; if $C = \mu I - A$, where $\mu > 0$, then C is nonsingular and $C^{-1} \geq [0]$ if and only if $\mu > \lambda^*(A)$. Moreover, $C^{-1} > [0]$ if and only if $\mu > \lambda^*(A)$ and A is indecomposable.

See also Aleskerov, Ersel and Piontovski (2011).

Fiedler and Ptàk (1962, 1966a, b) prove many other properties of $\mathcal{Z}$-matrices and of nonsingular $\mathcal{M}$-matrices (called $\mathcal{K}$-matrices by these authors). We point out the following ones.

I) Let be $A \in \mathcal{M}$, $B \in \mathcal{M}$, $AB \in \mathcal{Z}$. Then $AB \in \mathcal{M}$.

II) Let be $A \in \mathcal{M}$, $B \in \mathcal{Z}$, $AB \in \mathcal{M}$. Then $B \in \mathcal{M}$.

III) Let be $A \in \mathcal{M}$, $B \in \mathcal{Z}$, $B \geqq A$. Then:

i) $B \in \mathcal{M}$.

ii) $[0] \leqq B^{-1} \leqq A^{-1}$.

iii) $\det(B) \geqq \det(A) > 0$.

iv) $A^{-1}B \geqq I$; $BA^{-1} \geqq I$.

v) $B^{-1}A \leqq I$; $AB^{-1} \leqq I$.

vi) $B^{-1}A \in \mathcal{M}$; $AB^{-1} \in \mathcal{M}$.

vii) $w(B) \geqq w(A)$, where

$$w(A) = \min_{\lambda \in \sigma(A)} |\lambda| \text{ and } \sigma(A) = \{\lambda \in \mathbb{C} \ : |A - \lambda I| = 0\}.$$

viii) $1 - \rho(I - B^{-1}A) = 1 - \rho(I - AB^{-1}) = 1 / \rho(A^{-1}B) = 1 / \rho(BA^{-1})$, where

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

IV) Let be $C \in \mathcal{M}$ and let us consider two matrices A and B such that

$$B \geqq A \geqq C, \quad B \in \mathcal{Z}.$$

Then A, B, $B^{-1}C$ and $A^{-1}C$ are nonsingular $\mathcal{M}$-matrices and it holds

$$0 \leqq \rho(I - A^{-1}C) \leqq \rho(I - B^{-1}C) < 1.$$

We add some other "historical" considerations concerning the various characterizations of nonsingular $\mathcal{M}$-matrices. See also Berman and Plemmons (1994) and Plemmons (1977). Following McKenzie (1957, 1960), the characterization ($C4$) could be obtained directly from the results of Metzler (1945) and from the results of Hawkins and Simon, equivalent to ($C10$). This last characterization is quoted also by several economists, for example by Goodwin (1950), Chipman (1950), Solow (1952). The same characterization is also important as it defines, for square matrices (not necessarily $\mathcal{Z}$-matrices), the class of $\mathcal{P}$-matrices (see Section 5). The characterization ($C13$) has been proved by Varga (1962), as the related results of Ostrowski (1937) and Fan (1958) make reference to strongest results. The characterization ($C24$) can be directly obtained from the famous stability theorem of Lyapounov, popularized by the book of Gantmacher (1966). By the same criterion, it is easy to prove that the matrix $F = (-C)$, when C verifies ($C28$), is a stable matrix, even if $C \notin \mathcal{Z}$. See Arrow and McManus (1958), Quirk and Ruppert (1965) and Quirk and Saposnik (1968). Tartar (1971) has proved that ($C28$) is a necessary and sufficient condition for a $\mathcal{Z}$-matrix to be a nonsingular $\mathcal{M}$-matrix.

If the $\mathcal{Z}$-matrix C is represented in the form $C = (\lambda I - A)$, with $A \geqq [0]$, $\lambda > \lambda^*(A)$, then, by the characterization ($C13$), C is a nonsingular $\mathcal{M}$-matrix and C^{-1} admits the series expansion ("C. Neumann series")

$$C^{-1} = (\lambda I - A)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{+\infty} \frac{1}{\lambda^k} (A)^k,$$

where $(A)^k$ denotes the k-th power of A, and where $(A)^0 = I$. From the above series expansion it appears evident that, being $A \geqq [0]$, every element of $(\lambda I - A)^{-1}$ does not decrease if λ decreases, or if one or more elements of A increase and the other ones remain unchanged. This result is often used in the analysis of linear economic models; see, e. g., Debreu and Herstein (1953).

The characterization ($C44$) is taken from the paper of Gale and Nikaido (1965). Really, ($C44$) is equivalent to ($C10$) also when $C \notin \mathcal{Z}$. See also Nikaido (1968).

4. Proofs of the Equivalences

I) We begin by considering the basic paper of Fiedler and Ptåk (1962), which is one of the first papers which provides a complete path of the mathematical proofs of the various equivalent conditions considered by the said authors. The proof of the related theorem is rather short and elegant and can be evidenced by the following chain of implications.

$$\begin{array}{ccccccccc}
(C1) & \Longrightarrow & (C2) & \Longrightarrow & (C5) & \Longrightarrow & (C40) & \Longrightarrow & \\
\Longrightarrow & (C18) & \Longrightarrow & (C32) & & & & & \\
 & & & \Downarrow & & & & & \\
 & & & (C35) & \Longleftrightarrow & (C36) & & & \\
 & & & \Downarrow & & & & & \\
 & & & (C10) & \Longleftrightarrow & (C44) & & & \\
 & & & \Downarrow & & & & & \\
 & & & (C11) & \Longrightarrow & (C22) & \Longrightarrow & & \\
 & & \Longrightarrow & (C31) & \Longrightarrow & (C1). & & &
\end{array}$$

We add some comments on the above implications.

• $(C5) \Longrightarrow (C40)$.

The implication is proved by means of the concept of quasi-dominance of the matrix CD, with $D \in \mathcal{D}^+$.

• $(C40) \Longrightarrow (C18) \Longrightarrow (C32)$.

In the proof of the above implications the authors use the concept of "small matrix" (or "convergent matrix"), the theorem of Perron and Frobenius and the properties of the series of C. Neumann, of the type evidenced in $(C19)$ and $(C20)$.

• $(C35) \Longrightarrow (C10)$.

For the proof of the said implication the authors use a sufficient condition so that a $\mathcal{Z}$-matrix is a nonsingular $\mathcal{M}$-matrix. We remark that it is also possible to follow the method of proof suggested by Debreu and Herstein (1952).

• $(C11) \Longrightarrow (C22)$.

The proof is an immediate consequence of Theorems 3.1 and 3.2 of Fiedler and Ptàk (1962) and of the fact that it is always possible to write the $\mathcal{Z}$-matrix C as a product $C = RS$, with R a lower triangular matrix and S an upper triangular matrix, $R \in \mathcal{Z}^+$, $S \in \mathcal{Z}^+$.

• $(C35) \Longleftrightarrow (C36)$ and $(C10) \Longleftrightarrow (C44)$.

These coimplications are the unique directly proved by Fiedler and Ptàk. We have to note that the paper of Fiedler and Ptàk does not make distinctions between original characterizations and previous characterizations due to other authors. In particular we note that the equivalence $(C2) \Longleftrightarrow (C31)$ is due to Fan (1958).

II) The equivalences of Nikaido.

The books of Nikaido (1968, 1970) contain a proof of the equivalences evidenced in the following scheme.

$$\begin{array}{ccccccc}
(C9) & \Longrightarrow & (C4) & \Longrightarrow & (C3) & \Longrightarrow & (C9) \\
 & & \Updownarrow & & & & \\
 & & (C31) & \Longleftrightarrow & (C13) & &
\end{array}$$

The equivalence between $(C4)$ and $(C31)$ is proved by means of the well known result on the comparison between matrices ($A \geqq B$ if and only if $Ax \geqq Bx$, $\forall x \geqq [0]$). The equivalence between $(C31)$ and $(C13)$ is obtained by means of the Perron-Frobenius theorem. We have to note that Nikaido, as a mathematician involved in economic theory, gives some interesting comments on the above five equivalences. In particular, this author remarks the possibility of rewriting some characterizations in a "dual form", obtaining, for example, the equivalence between productivity and profitability in linear economic models with no joint productions (e. g. a Leontief model or a Sraffa model with simple production; see, e. g., Giorgi and Magnani (1978)).

III) The equivalence of Tartar.

Tartar (1971) proves the following equivalence

$$(C31) \iff (C28).$$

The result of Tartar is useful also for formulating new characterizations of nonsingular $\mathcal{M}$-matrices (see the point **V)** of the present section) and for establishing direct links with some characterizations, such as $(C27)$, $(C38)$ and $(C43)$.

IV) The equivalences of Varga.

The classical book of Varga (1962) contains various material on nonsingular $\mathcal{M}$-matrices, in particular the caracterizations $(C19)$, $(C31)$ and $(C37)$. His paper of 1976 (Varga (1976a)) contains several new characterizations, all proved by starting from $(C13)$. More precisely, this author proves that $(C13)$ is equivalent to:

$(C2)$, $(C3)$, $(C6)$, $(C7)$, $(C23)$, $(C52)$, , $(C53)$, $(C54)$ and $(C55)$.

One of the interesting feature of the paper of Varga (1976a) is the introduction of new extensions of the concept of diagonal dominance; for these questions see also Alefeld and Varga (1976), Beauwens (1976) and Varga (1976b).

V) Proof of the other equivalences.

We wish now to complete the "path" of the mathematical proofs of the main characterizations examined in the previous section. Obviously, in what follows $C \in \mathcal{Z}$.

• $(C12) \iff (C36)$.

The matrix C verifies $(C12)$ if and only if $(-C)$ is a stable matrix; the Routh-Hurwitz conditions are necessary and sufficient for the stability of a square matrix (not necessarily a $\mathcal{Z}$-matrix). As λ is an eigenvalue of C if and only if $-\lambda$ is an eigenvalue of $(-C)$, we have the said equivalence.

• $(C24) \iff (C36)$.

It is sufficient to follow the previous considerations, by noting that AC is quasi-positive definite if and only if $A(-C) = -(AC)$ is quasi-negative definite, and hence that $(C24)$ is nothing but the reformulation of the classical stability criterion of Lyapounov for the matrix $(-C)$. See, e. g., Gantmacher (1959), Quirk and Saposnik (1968).

• $(C39) \iff (C36)$.

Ostrowski (1937) has proved that if a square matrix C has a positive quasi-dominant diagonal, then it is a $\mathcal{P}$-matrix, i. e. all its principal minors are positive: This was shown also by McKenzie (1960). Hence if $C \in \mathcal{Z}$, then C is a nonsingular $\mathcal{M}$-matrix. Maybe it was McKenzie (1960) who first realized that the row quasi-dominance is equivalent to the column quasi-dominance and that every *Metzlerian matrix* A (i. e. $-A \in \mathcal{Z}$) is stable if and only if A has a negative quasi-dominant diagonal. This result allows to obtain at once the equivalence in question, by recalling that $(-C)$ has a negative quasi-dominant diagonal if and only if C has a positive quasi-dominant diagonal and that λ is an eigenvalue of C if and only if $(-\lambda)$ is an eigenvalue of $(-C)$.

• $(C27) \iff (C28)$.

If the $\mathcal{Z}$-matrix C verifies $(C28)$, then C verifies, with $E = I$, $(C27)$. If C verifies $(C27)$, then $F = CE \in \mathcal{Z}$ and verifies $(C28)$, as there exists $D \in \mathcal{D}^+$ such that $DF = DCE$ is quasi-positive definite. Therefore $F = DCE \in \mathcal{M}$ and also $C \in \mathcal{M}$. Therefore $(C27)$ holds.

• $(C25) \iff (C26) \iff (C24)$.

These equivalences are immediate, by recalling the definitions of a quasi definite matrix and of a definite matrix. Note that $(AC + C^\top A)$ is symmetric.

• $(C8) \iff (C47)$.

As the system $Cx \leqq [0]$ always admits a solution, as well as the system $x \geq [0]$, it is evident that $(C47)$ is equivalent to $(C8)$.

• $(C8) \iff (C45)$.

The same previous considerations hold.

• $(C47) \iff (C48)$.

If we put $y = -x$, then $Cx \leqq [0]$ is equivalent to $Cy \geqq [0]$ and $x \ngeq [0]$ is equivalent to $y \nleq [0]$. Hence $(C47)$ is equivalent to

$$Cy \geqq [0] \Longrightarrow y \nleq [0],$$

which is just $(C48)$.

• $(C51) \iff (C31)$.

If C verifies $(C31)$, it holds $C^{-1} \geq [0]$, hence $(DC)^{-1} = C^{-1}D^{-1}$, with $D \in \mathcal{D}$, and this product is a nonnegative matrix if and only if $D \in \mathcal{D}^+$. Hence from $(C31)$ we obtain $(C51)$. Vice-versa, if C verifies $(C51)$, then it must hold $C^{-1} \geq [0]$, i. e. $(C31)$.

• $(C46) \iff (C10)$.

We note that the class $\mathcal{M}$ can be redefined, thanks to $(C10)$, as the intersection between the $\mathcal{Z}$-matrices and the $\mathcal{P}$-matrices and that $(C46)$ characterizes the $\mathcal{P}$-matrices, thanks to a result of Gale and Nikaido (1965). The same reasoning holds for $(C10)$.

• $(C31) \iff (C50)$.

If C verifies $(C31)$, then C^{-1} exists and it holds $C^{-1} \geq [0]$, hence C verifies $(C50)$. Vice-versa, if C verifies $(C50)$, C^{-1} exists and from $(C50)$, with $y = e^k$ (k-th unit or standard vector of $\mathbb{R}^n$), we have $C^{-1}e^k = (C^{-1})^k \geq [0]$. Hence $(C31)$ is verified.

In order to prove the next results, the following remarks are useful, even if trivial.

Remark 1. Given the split $C = (\mu I - A)$, we have $A = (\mu I - C)$.

Remark 2. If $C \in \mathcal{Z}$, in the split $C = (\mu I - A)$ we have $A \geqq [0]$ if

$$\mu \geqq \max_{i \in N} \{c_{ii}\}, \ N = \{1, 2, ..., n\}.$$

See also what said in Section 3, after the characterization $(C13)$.

Remark 3. The number α is an eigenvalue of $A = (\mu I - C)$ if and only if $(\mu - \alpha)$ is an eigenvalue of C. Indeed, with $C = (\mu I - A) = \mu I - (\mu I - C)$, we have $(A - \alpha I) = \mu I - C - \alpha I = -[C - (\mu - \alpha)I]$. Hence, $\det(A - \alpha I) = 0 \iff \det(C - (\mu - \alpha)I) = 0$.

• $(C16) \iff (C35)$.

Let $(C35)$ hold. We first note that, with $C \in \mathcal{Z}$, it is possible to obtain the split of $(C16)$: it is sufficient (Remarks 1 and 2) to choose μ as in Remark 2. If $\mu \leqq \lambda^*(A)$, being $\lambda^*(A)$ an eigenvalue of A, by Remark 3, $(\mu - \lambda^*(A))$ is an eigenvalue of C and hence this matrix has a nonpositive eigenvalue, in contrast with $(C35)$. Hence $(C35)$ implies $(C16)$. Vice-versa, if $(C16)$ holds, there exist infinite splits of C which verify the first member of $(C16)$ and hence also the second member will be verified.

• $(C14) \iff (C13)$.

Trivial, thanks to Remark 1.

• $(C17) \iff (C18)$.

Trivial, thanks to Remarks 1 and 2.

$(C15) \iff (C14)$.

Trivial, thanks to Remarks 1 and 2.

•$(C21) \iff (C1)$.

If C verifies $(C1)$, then C verifies also $(C21)$, by choosing $B = C$ and $A = [0]$. With these choices we have $B - A = C - [0] = C$, $B = C \in \mathcal{S}$ (thanks to the definition of the $\mathcal{S}$-class), C nonsingular (by the equivalence between $(C1)$ and $(C31)$), $B^{-1}A = C^{-1}[0] = [0]$, and hence $B^{-1}A$ small matrix, as a zero square matrix has all its eigenvalues equal to zero. Now suppose that the $\mathcal{Z}$-matrix C verifies $(C21)$. Then $B^{-1}C = B^{-1}(B - A) = I - B^{-1}A$. But, being $B^{-1}A \geqq [0]$, it holds $B^{-1}C \in \mathcal{Z}$. Being $B^{-1}A$ a small matrix, we have also (by $(C13)$, with 1 instead of μ and $(B^{-1}A)$ instead of A) that $B^{-1}A \in \mathcal{M}$. Then, by $(C31)$, we have $(B^{-1}C)^{-1} = C^{-1}B \geqq [0]$. If $B \in \mathcal{S}$, then there exists a vector $q \geqq [0]$ such that $Bq > [0]$, i. e. $CC^{-1}Bq > [0]$, i. e. $C(C^{-1}Bq) > [0]$. But being $q \geqq [0]$ and $C^{-1}B \geqq [0]$, we have also that $\bar{q} = C^{-1}Bq \geqq [0]$. Hence there exists $\bar{q} \geqq [0]$ solution of $C\bar{q} > [0]$, i. e. $(C1)$ holds. If it results $A \in \mathcal{S}$, then there exists $x \geqq [0]$ such that $Ax > [0]$, i. e. $C[(C^{-1}B)(B^{-1}A)]x > [0]$. If we put $\bar{x} = (C^{-1}B)(B^{-1}A)x$, being x, $C^{-1}B$ and $B^{-1}A$ nonnegative, we have that there exists $\bar{x} \geqq [0]$ solution of $C\bar{x} > [0]$, i. e. also in this case $(C1)$ holds.

• $(C1) \iff (C29)$.

If the $\mathcal{Z}$-matrix C verifies $(C1)$, then it verifies also the equivalent proposition $(C28)$: there exists a matrix $A \in \mathcal{D}^+$ which makes AC quasi-positive definite, i. e. $(AC + C^\top A)$ positive definite. Being $A \in \mathcal{D}^+$, we have $AC \in \mathcal{Z}$, $(AC)^\top = C^\top A \in \mathcal{Z}$ and $(AC + C^\top A) \in \mathcal{Z}$. By a result of Fiedler and Ptàk (1966b), i. e.

$$B \text{ quasi-positive definite} \Longrightarrow B \in \mathcal{P} \Longrightarrow B \in \mathcal{S},$$

we have that $(AC + C^\top A) \in \mathcal{S}$, as required by $(C29)$.

Vice-versa, if C verifies $(C29)$, the $\mathcal{Z}$-matrix $(AC + C^\top A)$ is also an $\mathcal{S}$-matrix. Hence it is a nonsingular $\mathcal{M}$-matrix, as it verifies $(C35)$. But being a symmetric matrix, all its eigenvalues are positive. hence, if $(C24)$ holds, then $(C26)$ holds. The equivalences $(C26) \iff (C24)$ and $(C24) \iff (C36)$ have been already proved and hence $(C36)$ holds, together with its equivalent proposition $(C1)$.

• $(C1) \iff (C30)$.

If C verifies $(C30)$, then it verifies also $(C1)$, as just proved. Vice-versa, if C verifies $(C1)$, we obtain at once proposition $(C30)$, following the same proof of the previous equivalence.

• $(C31) \iff (C34)$.

If C verifies $(C31)$, then, with $R = S = I$, we have (RCS) nonsingular and $(RCS)^{-1} = C^{-1} \geq [0]$, and hence $(C34)$ holds. If $(C34)$ holds, by the existence and nonnegativity of $(RCS)^{-1}$, we have that RC and S are nonsingular and we have that there exists $Q \geqq [0]$ such that $(RCS)^{-1} = Q$. From this relation we have

$$S^{-1}C^{-1}R^{-1} = Q; \quad S(S^{-1}C^{-1}R^{-1})R = SQR; \quad C^{-1} = SQR.$$

From the last relation we have $C^{-1} \geq [0]$, being $Q \geqq [0]$, $R \geqq [0]$ and $S \geqq [0]$, hence $(C31)$ holds.

• $(C1) \iff (C8)$.

Let us suppose that $C \in \mathcal{Z}$ verifies $(C1)$. Then, also $C^\top$ verifies $(C1)$, i. e. the system $C^\top x > [0]$ admits a solution $x \geqq [0]$. By the *theorem of the alternative of Ville* (see, e. g., Cottle, Pang and Stone (2009), Gale (1960), Mangasarian (1969)), the system $Cx \leqq [0]$, $x \geq [0]$ does not admit a solution. Therefore from $(C1)$ it follows $(C8)$. Vice-versa, if $(C8)$ holds, for the same theorem of the alternative, there exists a solution $q \geqq [0]$ of the system $C^\top q > [0]$, hence $C^\top$ verifies $(C1)$, but then also C verifies $(C1)$.

• $(C31) \iff (C49)$.

We recall the following result (see, e. g., Nikaido (1970), theorem 15.1): if A and B are two real matrices of the same order, then it holds $A \geqq B$ if and only if $Ax \geqq Bx$, $\forall x \geqq [0]$. If C verifies $(C31)$, then $C^{-1} \geq [0]$, hence $C^{-1}q \geqq [0]$, $\forall q \geqq [0]$, i. e. the implication $q \geqq [0] \Longrightarrow C^{-1}q \geqq [0]$ holds. Then we have

$$C(C^{-1}q) \geqq [0] \Longrightarrow (C^{-1}q) \geqq [0].$$

If we put $x = C^{-1}q$, we have a ono-to-one transformation which allows to write

$$Cx \geqq [0] \Longrightarrow x \geqq [0],$$

i. e. $(C49)$.

Vice-versa, let us suppose that $(C49)$ holds. We remark that if there exists a vector h such that $Ch = [0]$, then it will hold also $C(-h) = [0]$ and from $(C49)$ we get $h \geqq [0]$ and

$(-h) \geqq [0]$, i. e. $h = [0]$. Hence, $(C49)$ implies that C is nonsingular, which allows to rewrite $(C49)$, with $x = C^{-1}q$, in the form

$$C(C^{-1}q) \geqq [0] \Longrightarrow C^{-1}q \geqq [0]\,,$$

i. e.

$$q \geqq [0] \Longrightarrow C^{-1}q \geqq [0]\,, \text{ i. e. } C^{-1}q \geqq [0]\,, \ \forall q \geqq [0]\,.$$

Hence we get $C^{-1} \geq [0]$, i. e. $(C31)$.

• $(C28) \iff (C38)$.

If C verifies $(C28)$, then there exists a diagonal matrix $D \in \mathcal{D}^+$ such that DC is quasi-positive definite; thanks to results of Johnson (1974), condition $(C38)$ holds. Vice-versa, from $(C38)$ we get, with $P = I$, that the matrix C has all its eigenvalues with a positive real part, i. e. C veifies $(C35)$ and hence also its equivalent condition $(C28)$.

• $(C10) \iff (C41)$.

If C verifies $(C10)$, then C is a $\mathcal{P}$-matrix and by a result of Fiedler and Ptàk (1966b), C verifies $(C41)$. Thanks to the same result, which characterizes the $\mathcal{P}$-class, we have that $(C41)$ implies $(C10)$.

• $(C10) \iff (C42)$.

The previous considerations hold also for this case.

• $(C33) \iff (C32)$.

The characterization $(C33)$ is due to Poole and Boullion (1974) who prove its equivalence to $(C32)$, due to Fiedler and Ptàk (1962). We have to observe that Poole and Boullion are concerned with nonsingular $\mathcal{M}$-matrices which belong to the $\mathcal{Z}^+$-class. Within this class the proposition of Poole and Boullion

$$W \geqq C, \ C \in \mathcal{Z}^+ \Longrightarrow \det(W) \neq 0$$

is indeed a characterization of nonsingular $\mathcal{M}$-matrices. We have adopted the form $(C33)$, just in order to take into account of $\mathcal{Z}$-matrices for which we have not a priori informations on the sign of the main diagonal elements. The equivalence of the present point is immediate, by the fatc that $C \in \mathcal{Z}^+$ is a necessary condition such that $C \in \mathcal{M}$.

5. Some Connections and Extensions

We give in the present section only some hints on the various classes of matrices connected to nonsingular $\mathcal{M}$-matrices or that are a generalization of this class. For further considerations (the literature is indeed abundant) we refer the reader to the basic papers of Fiedler and Ptàk (1962, 1966ab, 1967) and Johnson (1974). Moreover: Berman and Plemmons (1994), Giorgi and Zuccotti (2009, 2014, 2015a,b).

In the present paper we have taken into consideration *nonsingular* $\mathcal{M}$-matrices, however, also *singular* $\mathcal{M}$-matrices have been introduced by various authors, as a generalization of the first class of matrices. Some authors speak also of "general $\mathcal{M}$-matrices". See, e. g., Berman and

Plemmons (1994), Neumann and Plemmons (1980) and Poole and Boullion (1974). Singular $\mathcal{M}$-matrices have many applications, similarly to nonsingular $\mathcal{M}$-matrices, however singular $\mathcal{M}$-matrices are more difficult to study.

If M is a square (real) matrix of order n, with $M \in \mathcal{Z}$, then each of the following conditions is equivalent to the statement "M is a singular $\mathcal{M}$-matrix".

a) All principal minors of M are nonnegative.

b) Every real eigenvalue of each principal submatrix of M is nonnegative.

c) $M + D$ is nonsingular for each positive diagonal matrix D.

d) For each $x \neq [0]$ there exists a nonnegative diagonal matrix D such that

$$x^\top Dx \neq 0 \text{ and } x^\top MDx \geqq 0.$$

e) The sum of all the $k \times k$ principal minors D_k of M is nonnegative for $k = 1, ..., n$.

f) Every real eigenvalue of M is nonnegative.

g) $M + \alpha I$ is nonsingular for each $\alpha > 0$.

h) The real part of each nonzero eigenvalue of M is positive.

i) M is *nonnegative stable,* i. e. the real part of each eigenvalue of M is nonnegative.

l) The matrix M can be represented in the form

$$M = sI - A,$$

where $A \geqq [0]$ and $s \geqq \lambda^*(A)$.

We have already introduced (see the characterization $(C36)$) the class of *Metzler matrices* or *Metzlerian matrices,* as those square matrices F such that $-F \in \mathcal{Z}$, i. e. $f_{ij} \geqq 0$, $\forall i \neq j$. Hence, the class of Metzler matrices is closely related to the class of nonsingular $\mathcal{M}$-matrices. Metzlerian matrices are important in the study of stability conditions for a Walrasian equilibrium market. See, e. g., Giorgi (2003), Kemp and Kimura (1978), Quirk and Saposnik (1968), Takayama (1985), Woods (1978). On the grounds of the previous results, the following properties are immediate.

Theorem 3. Let A be a Metzler matrix of order n. Then the following conditions are equivalent.

1) $A = M - \alpha I$, where $M \geqq [0]$ and $\alpha > \lambda^*(A)$.

2) A is a stable matrix, i. e. $\mathrm{Re}(\lambda) < 0$, for each eigenvalue λ of A.

3) A is D-stable, i. e. DA is stable for every $D \in \mathcal{D}^+$.

4) A is totally stable, i. e. every principal submatrix of A is D-stable.

5) Every principal minor of A has the sign of $(-1)^i$, $i = 1, ..., n$. This property, in economic analysis, is described also by saying that A is "Hicksian", in honour of the English economist J. Hicks. Hicksian matrices are also called "$\mathcal{NP}$-matrices".

6) The leading principal minors of A have the sign of $(-1)^i$, $i = 1, ..., n$.

7) A has a negative quasi-dominant diagonal

8) A^{-1} exists and it holds $A^{-1} \leq [0]$.

9) There exists a vector $x > [0]$ such that $Ax < [0]$.

Economists have tried to enlarge the class of Metzlerian matrices, in order to find more general conditions assuring the stability of multiple competitive markets. A powerful generalization of Metzlerian matrices is given by the so-called *Morishima matrices.* See, e. g., Giorgi and Zuccotti (2015b), Kemp and Kimura (1978), Quirk and Saposnik (1968). See also Bassett, Habibagahi and Quirk (1967) and Quirk (1974).

Definition 3. A square (real) matrix A of order n is said to be a *Morishima matrix* if there exist subsets J and K of $N = \{1, 2, ..., n\}$ such that:

i) $J \cap K = \varnothing$;

ii) $J \cup K = N$;

iii) if $i \neq j$, then

$$\begin{aligned} a_{ij} &\geqq 0, \text{ for } i, j \in J \text{ or } i, j \in K, \\ a_{ij} &\leqq 0, \text{ otherwise.} \end{aligned}$$

By the previous definition we have that A is a Morishima matrix if it can be written in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a Metzlerian matrix of order r, $0 \leqq r \leqq n$, A_{22} is a Metzlerian matrix of order $(n-r)$, and all entries in A_{12} and A_{21} are nonpositive. A can also be written as $A = I^* M I^*$, where

$$I^* = \begin{bmatrix} -I_1 & [0] \\ [0] & I_2 \end{bmatrix},$$

with I_1 identity matrix of order r and I_2 identity matrix of order $(n-r)$ and where M is a Metzlerian matrix.

Theorem 4. Let A be a Morishima matrix of order n. Then the following conditions are equivalent.

1) $A = I^*(M - \alpha I)I^*$, where $M \geqq [0]$ and $\alpha > \lambda^*(M)$.

2) A is a stable matrix.

3) A is a D-stable matrix.

4) A is a totally stable matrix.

5) A is Hicksian.

6) The leading principal minors of A have the sign of $(-1)^i$, $i = 1, ..., n$.

7) A has a negative quasi-dominant diagonal.

8) A^{-1} exists, with

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

the partitionning corresponding to that of A, with all entries in B_{11} and B_{22} being nonpositive, and all entries in B_{12} and B_{21} being nonnegative (obviously A^{-1} has every line different from the zero vector).

9) There exists a vector $x = [x^1,\ x^2]^\top$, where $x^1 < [0]$, $x^1 \in \mathbb{R}^r$, $x^2 > [0]$, $x^2 \in \mathbb{R}^{n-r}$, such that $Bx = y$, $y = [y^1,\ y^2]^\top$, $y^1 > [0]$, $y^1 \in \mathbb{R}^r$, $y^2 < [0]$, $y^2 \in \mathbb{R}^{n-r}$.

Another class of square matrices related to nonsingular $\mathcal{M}$-matrices is the class of $\mathcal{P}$-matrices. This class was considered (with this name) by Fiedler and Ptàk (1962) and by Gale and Nikaido (1965), these last authors in order to establish for functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a *global univalence theorem.* However, the concept of a $\mathcal{P}$-matrix finds other useful applications, mainly in the study of the stability conditions for a multiple exchange market. More precisely, the so-called "Hicks perfect stability conditions" are given in terms of the signs of the principal minors of the Jacobian of the excess demand functions: the negative of the said Jacobian must be a $\mathcal{P}$-matrix. As previously said in Theorem 3, in economic analysis it is said that the said Jacobian is an "Hicksian matrix".

Definition 4. A square matrix A of order n is said to be a $\mathcal{P}$-matrix if all its principal minors are positive.

We have the following results on $\mathcal{P}$-matrices (see, e. g., Bapat and Raghavan (1997), Berman and Plemmons (1994), Nikaido (1968), Kemp and Kimura (1978), Fiedler and Ptàk (1962, 1966b), Woods (1978)).

Theorem 5. Let A be a square matrix of order n. Then the following conditions are equivalent.

i) A is a $\mathcal{P}$-matrix.

ii) Every real eigenvalue of each principal submatrix of A is positive.

iii) For each $x \neq [0]$ there exists a positive diagonal matrix D such that

$$x^\top ADx > 0.$$

iv) For each $x \neq [0]$ there exists a nonnegative diagonal matrix D such that

$$x^\top ADx > 0.$$

v) The matrix A "reverses the sign" of the zero vector only, i. e.

$$\{x_i(A_i x) \leqq 0,\ \forall i \in \{1, ..., n\}\} \Longrightarrow x = [0]\,.$$

vi) For each *signature matrix* S (i. e. S is diagonal with diagonal entries ± 1), there exists an $x > [0]$ such that

$$SASx > [0]\,.$$

A sufficient condition for A to be a $\mathcal{P}$-matrix is that A has a positive quasi-dominant diagonal (see. e. g., McKenzie (1960)). As previously remarked, $\mathcal{P}$-matrices have been used to obtain a global univalence theorem for functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Theorem 6 (Gale and Nikaido). If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable mapping on the open multi-dimensional interval $I \subset \mathbb{R}^n$, and the Jacobian $Jf(x)$ is a $\mathcal{P}$-matrix for all $x \in I$, then f is univalent on I.

See also Garcia and Zangwill (1979) and Parthasarathy (1983). Another important field of applications of $\mathcal{P}$-matrices is the theory of *Linear Complementarity Problems* (LCP). See the basic book of Cottle, Pang and Stone (2009).

Given a vector $r \in \mathbb{R}^n$ and a (real) square matrix M of order n, find (if possible) $z \in \mathbb{R}^n$ such that

$$w = r + Mz,$$

with $w \geqq [0]\,,\ z \geqq [0]\,,\ z^\top w = 0$.

A closely related class of $\mathcal{P}$-matrices is the class of $\mathcal{P}_0$-matrices, introduced by Fiedler and Ptàk (1966b) and by Arrow (1974) in his analysis of the stability of a competitive equilibrium. This class of square matrices represents the "closure" of the class of $\mathcal{P}$-matrices.

Definition 5. A square matrix A of order n is said to be a $\mathcal{P}_0$-*matrix* or to belong to the $\mathcal{P}_0$-*class,* if all its principal minors are nonnegative. The following equivalent characterizations of the $\mathcal{P}_0$-class hold.

Theorem 7. Let A be a square matrix of order n. Then the following properties are equivalent.

1) All principal minors of A are nonnegative, i. e. $A \in \mathcal{P}_0$.

2) For each vector $x \neq [0]$ there exists an index k such that $x_k \neq 0$ and $x_k y_k \geqq 0$, where $y = Ax$.

3) For each vector $x \neq [0]$ there exists $D_x \in \mathcal{D}$, $D_x \geqq [0]\,$, such that $x^\top D_x x > 0$ and $x^\top A^\top D_x x \geqq 0$.

4) Every real eigenvalue of A as well as of each principal minor of A is nonnegative.

5) $(A + \varepsilon I) \in \mathcal{P}$, $\forall \varepsilon > 0$.

6) For any $D \in \mathcal{D}^+$ (D of order n), (A+D) is a $\mathcal{P}$-matrix.

7) For any $D \in \mathcal{D}^+$ (D of order n), every real eigenvalue of DA is nonnegative.

Obviously, if $A \in \mathcal{P}$, then $A^\top \in \mathcal{P}$; if $A \in \mathcal{P}_0$, then $A^\top \in \mathcal{P}_0$.

Moreover, if $A + A^\top$ is positive definite (i. e. A is quasi-positive definite), then $A \in \mathcal{P}$. If $A + A^\top$ is positive semidefinite, then $A \in \mathcal{P}_0$.

Another class of square matrices related to $\mathcal{M}$-matrices is the class of matrices with a dominant diagonal or with a quasi-dominant diagonal, previously recalled in Section 2. Besides the definitions given in Section 2, there are in the specialistic literature several other definitions of matrices with a dominant diagonal. We quote only Varga (1976b), Beauwens (1976), De Giuli, Magnani and Moglia (1994), Pearce (1974), Okuguchi (1976, 1978), Giorgi and Zuccotti (2009), Magnani (1972-73). See also Kemp and Kimura (1978). We wish here to make some considerations on the contribution of Fiedler and Ptàk (1967), not often considered in the literature. These authors consider two different notions of dominant diagonal matrices:

1) The square matrix A of order n has a *strong* dominant diagonal in the sense of Fiedler and Ptàk if there exists a matrix $D \in \mathcal{D}$ such that $B = D^{-1}AD$ has a row dominant diagonal (in the sense of Hadamard).

The above class of matrices coincides with the class of $\mathcal{H}$-*matrices* introduced by Ostrowski (1956).

2) The square matrix A of order n has a *weak* dominant diagonal in the sense of Fiedler and Ptàk if there exists a matrix $D \in \mathcal{D}$ such that for $B = D^{-1}AD$ it holds

$$i \neq j \Longrightarrow |b_{ii}| > |b_{ij}| \, .$$

This second definition is introduced by Fiedler and Ptàk in order to define the class of the $\mathcal{W}$*-matrices,* which form the central subject of their paper. We shall not be concerned with the above second definition. As for what concerns the first definition, Fiedler and Ptàk present a result which can be genaralized, in obtaining the following theorem.

Theorem 8. Let be $A \in \mathcal{Z}$. Then $A \in \mathcal{M}$ if and only if there exist two diagonal matrices $D, E \in \mathcal{D}^+$ such that DAE has a positive row (column) quasi-dominant diagonal.

The previous theorem is obtained by the characterization ($C39$), recalling that a matrix has a row quasi-dominant diagonal if and only if it has a column quasi-dominant diagonal and recalling Theorem 1 (transformation $f_2(C)$).

The results of Fiedler and Ptàk (1967) suggest the possibility to introduce another definition of matrices with a dominant diagonal, matrices we shall call "matrices with *general* dominant diagonal".

Definition 6. The square matrix A of order n has a *general dominant diagonal* if there exist two matrices $D, E \in \mathcal{D}^+$, such that, with $T = DZ_AE$ ($T = D(Z_A)^\top E$), being Z_A the *comparison matrix* of A, it holds

$$Te \geqq [0]\,; \tag{4}$$

$$Te > U(T)e. \tag{5}$$

On the above definition the following remarks may be useful.

Remark 4.

i) In Definiiton 6 the properties of T are equivalent to the fact that T is a matrix with dominant diagonal in the sense of Beauwens (1976) and Varga (1976b).

ii) Every matrix with a dominant diagonal is a matrix with a general dominant diagonal; indeed, with $D = E = I$ we have either $Te > [0]$ or $T^\top e > [0]$ and, being $U(T) \leqq [0]$, (4) and (5) both hold.

iii) A matrix has a general dominant diagonal if and only if it has a quasi-dominant diagonal (in the sense of McKenzie). Indeed, if A has a row (column) quasi-dominant diagonal, there exists $E \in \mathcal{D}^+$ ($D \in \mathcal{D}^+$) such that $Z_AEe > [0]$ ($e^\top DZ_A > [0]$) and Z_A has a positive diagonal. Hence, with $D = I$ ($E = I$) it holds, with $T = DZ_AE$ ($T = D(Z_A)^\top E$), $Te > [0]$, i. e. (4) holds. Being $D, E \in \mathcal{D}^+$, we have that $T \in \mathcal{Z}$ and in the decomposition

$$T = D_T + U_T + L_T = D(T) + U(T) + L(T)$$

we have $U_T \leqq [0]$ and $L_T \leqq [0]$. This gives $U_Te \leqq [0]$ and from $Te > [0]$ we get relation (5). Viceversa if A has a general dominant diagonal, (4) and (5) hold, i. e. by characterization ($C7$), the

$\mathcal{Z}$-matrix T is a nonsingular $\mathcal{M}$-matrix. As $D, E \in \mathcal{D}^+$, and the classes of $\mathcal{Z}$-matrices and $\mathcal{M}$-matrices are closed with respect to the transformation $f(Z_A) = DZ_AE$, we have that $Z_A \in \mathcal{M}$ and, by characterization $(C39)$ we have that Z_A has a positive quasi-dominant diagonal, i. e. A has a quasi-dominant diagonal.

iv) In a similar way it is possible to prove that A has a general dominant diagonal if and only if either A or $A^\top$ has a strong dominant diagonal in the sense of Fiedler and Ptàk (1967). See, in particular their theorem 1.2.

v) In conclusion, the notion of general diagonal dominance is substantially equivalent to the quasi-diagonal dominance in the sense of McKenzie and to the strong diagonal dominance in the sense of Fiedler and Ptàk.

Another class of square matrices related to nonsingular $\mathcal{M}$-matrices is the class of *inverse-positive matrices,* i. e. those square matrices A for which A^{-1} exists and $A^{-1} \geq [0]$, in the sense that A^{-1} has all semipositive lines (recall the characterization $(C31)$ of nonsingular $\mathcal{M}$-matrices). Perhaps it would be more correct to speak of "inverse-semipositive matrices". We have the following result (see, e. g., Berman and Plemmons (1994), Plemmons (1977)).

Theorem 9. Let A be a square matrix of order n. Then the following conditions are equivalent.

a) A is *inverse-positive,* i. e. A^{-1} exists and

$$A^{-1} \geq [0] .$$

b) A is *monotone,* i. e.

$$Ax \geqq [0] \Longrightarrow x \geqq [0] , \ \forall x \in \mathbb{R}^n.$$

c) There exists an inverse-positive matrix $B \geqq A$ such that $I - B^{-1}A$ is convergent, i. e. $\rho(I - B^{-1}A) < 1$.

d) There exist inverse-positive matrices B and C such that

$$B \leqq A \leqq C.$$

e) There exists an inverse-positive matrix $B \geqq A$ and a nonsingular $\mathcal{M}$-matrix C, such that

$$A = BC.$$

f) There exists an inverse-positive matrix B and a nonsingular $\mathcal{M}$-matrix C, such that

$$A = BC.$$

g) A has a *convergent regular splitting,* that is A has the following representation:

$$A = M - N, \ M^{-1} \geq [0] , \ N \geqq [0]$$

where $M_{-1}N$ is convergent.

h) A has a *convergent weak regular splitting,* that is A has the following representation:

$$A = M - N, \; M^{-1} \geq [0] \quad M^{-1}N \geqq [0]$$

where $M^{-1}N$ is convergent.

Another result on inverse-positive matrices is given by the following theorem, perhaps more admitting of economic interpretations. See, e. g., Abad, Gassò and Torregrosa (2011).

Theorem 10. The real square matrix A of order n is inverse-positive if and only if for all $y > [0]$ there exists $x > [0]$ such that $Ax = y$.

Fujimoto and Ranade (2004) have provided a necessary condition for the inverse-positivity of matrices by means of a generalization of the *Hawkins-Simon conditions.* Other related results are in the papers of Bidard (2007), Eisner (Recte: Elsner), Olesky and van den Driessche (2009), Fiedler and Grone (1981), Johnson (1983), Johnson, Leighton and Robinson (1979).

Obviously, the class of inverse-positive matrices contains the class of $\mathcal{M}$-matrices, hence the analysis of the class of inverse-positive matrices can be useful to study, e. g., those linear economic models not described by a matrix $A \in \mathcal{Z}$. It is the case, for example, of linear models with joint production, such as some models of P. Sraffa and the growth economic model of J. von Neumann. See, e. g., Giorgi and Magnani (1978), Kurz and Salvadori (1995), Schefold (1989), Peris and Villar (1993). For the von Neumann model see Murata (1977), Nikaido (1968, 1970), Takayama (1985), Woods (1978). These models are usually described by two semipositive matrices A and B, not necessarily square, where A is the matrix of the *inputs* and B is the matrix of the *outputs.* If A and B are *square,* as in the models considered by P. Sraffa, and if it happens that $(B - A)^{-1} \geq [0]$, economists speak of "all-productive" models, if $(B - A)^{-1} > [0]$, of "all-engaging" models (see, e. g., Schefold (1989), Giorgi (2014)).

All-productive models (and all-engaging models) have several properties of single production models, therefore they are easier to be analyzed, also form a matehmatical point of view. See Schefold (1978). Obviously, if $(B - A) \in \mathcal{Z}$, then if any of the conditions of Section 3 holds, it holds $(B - A)^{-1} \geq [0]$, i. e. the model is all-productive. Otherwise, Theorems 9 and 10 apply. We have however to remark that conditions *g)* and *h)* of Theorem 9 are not too convenient for characterizing all-productive economic models, where "effective" joint production is considered, i. e. B is not the identity matrix or a diagonal matrix. Indeed, a result of Johnson (1983) states that a semipositive square matrix has a nonnegative (i. e. a semipositive) inverse only if it is a diagonal matrix or a permutation of a diagonal matrix, and hence "effective" joint production is ruled out by the above conditions *g)* and *h)* of Theorem 9. We have to look for other conditions. The following result of Peris (1991) is quite interesting; see also Giorgi (2014), Peris and Villar (1993).

Definition 7. A split of a square matrix $M = B - A$, where $A \geqq [0]$, $B \geqq [0]$, is called a *positive split.* A positive split is said to be a *B-split* if B is nonsingular and

a) $Bx \geqq [0] \Longrightarrow Ax \geqq [0]$,

$b)$ For all $x \in \mathbb{R}^n$ it holds

$$\begin{pmatrix} M \\ B \end{pmatrix} x \geqq [0] \Longrightarrow x \geqq [0] \, .v$$

Notice that any $\mathcal{Z}$-matrix has a B-split, but the converse is not true. Condition $a)$ in Definition 7 is equivalent to the existence of a nonnegative square matrix H, of order n, such that $A = HB$, i. e., being B nonsingular by assumption, to $AB^{-1} \geqq [0]$. See Mangasarian (1971). This condition (the input matrix cone contains the output matrix cone) is also introduced by the English economist J. Hicks (1965) and considered, subsequently, also in the analysis of the celebrated von Neumann growth model. See, e. g., Giorgi (2016), Łos (1971), Thompson and Weil (1971). We note, moreover, that, always under the above condition, the Perron-Frobenius results apply for the problem $Ax = \lambda Bx$; see Mangasarian (1971).

Theorem 11. Let M be a square matrix such that $M = B - A$ is a B-split. Then the following conditions are equivalent:

(a) M is inverse-positive (i. e. the joint production economic model described by the pair (A, B) is all-productive).

(b) $\lambda^*(AB^{-1}) < 1$, i. e. AB^{-1} is small (or convergent).

(c) There exists some $x \geq [0]$ such that $Mx > [0]$ (i. e. the joint production economic model described by the pair (A, B) is *productive*).

We have mentioned the assumption that in a joint production model described by the pair (A, B), $A \geqq [0]$, $B \geqq [0]$, both square of order n, we have $(B - A) \in \mathcal{Z}$. Indeed, this convenient assumption is adopted by some authors dealing with joint production Sraffa's models. However, in our opinion, its meaning, from a pure economic point of view, is doubtful. It must be $b_{ij} \leqq a_{ij}$, $\forall i \neq j$, i. e. the off-diagonal elements of the output matrix B must be less or equal than the corresponding elements of the input matrix A. If this is meaningful when $B = I$, in case we have an "effective" joint production, this is perhaps less meaningful. Perhaps it is more economically meaningful to adopt the opposite assumption: $(A - B) \in \mathcal{Z}$, i. e. $(B - A)$ is a *Metzlerian matrix,* i. e. $b_{ij} \geqq a_{ij}$, $\forall i \neq j$. Under this assumption we have a result, due to Buffoni and Galati (1974), concerning the inverse-positivity of $(B - A)$.

Let M be a nonsingular indecomposable Metzlerian matrix of order n (the authors call "essentially positive" such a matrix) and let be $\Lambda_{(i)}$ the matrix of order $(n-1)$ obtained from M by deleting its i-th row and its i-th column. The matrix $\Lambda_{(i)}$ has a real eigenvalue μ_i greater than the real part of all other eigenvalues (see Varga (1962)).

Theorem 12. Under the above assumptions, necessary and sufficient conditions to have $M^{-1} > [0]$ are:

(a) For some index i_0, $1 \leqq i_0 \leqq n$, it holds $\mu_{i_0} < 0$;

(b) For every $i = 1, ..., n$, it holds

$$\frac{\det(\Lambda_{(i)})}{\det(A)} > 0.$$

In particular, under the above assumptions, if $M^{-1} > [0]$, we have

$$(-1)^n \det(M) < 0; \quad m_{ii} < 0, \ i = 1, ..., n.$$

In Giorgi and Magnani (1978) there is the following example of $M = (B - A)$, with M Metzlerian and $M^{-1} > [0]$:

$$M = B - A = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 2 & 2 \\ 5 & 3 & 2,5 \end{bmatrix} - \begin{bmatrix} 1,5 & 1 & 4 \\ 0 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} -1,5 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1,5 \end{bmatrix}.$$

Obviously, if $M = B - A$, with $A, B \geqq [0]$, satisfies the assumptions under which Theorem 12 holds, then the joint production model described by (A, B) is all-engaging.

In the next extensions, due to Fiedler and Ptàk (1966b), the related matrices are not necessarily square.

Definition 8. A matrix A of order (m, n) is an $\mathcal{S}$-*matrix* or a *matrix of class* $\mathcal{S}$, if the system

$$\begin{cases} Ax > [0] \\ x \geqq [0] \end{cases}$$

admits a solution x.

We have previously remarked in Section 2 that equivalently a matrix A is in $\mathcal{S}$ if and only if the system $Ax > [0]$ has a solution $x \geq [0]$ or also $x > [0]$. The name of this class of matrices derives from E. Stiemke, one of the first mathematicians to recognize the importance of positivity in linear systems. See Stiemke (1915). Fiedler and Ptàk (1966b) prove the following result.

• The matrix A, of order (m, n), is an $\mathcal{S}$-matrix if and only if for every vector $p \geq [0]$, $p \in \mathbb{R}^m$, at least one component of $p^\top A$ is positive.

If A is *square,* then a characterization of A as an $\mathcal{S}$-matrix is related to the feasibility of a *linear complementarity problem*; see, e. g., Cottle, Pang and Stone (2009). Economic applications of the class of $\mathcal{S}$-matrices will be discussed in the next section. In case $A \in \mathcal{S}$ and A is *square,* some authors call A "semipositive", but we prefer to avoid this term, which may generate confusions.

Definition 9. A matrix A of order (m, n) is an $\mathcal{S}_0$-*matrix* if the system

$$\begin{cases} Ax \geqq [0] \\ x \geq [0] \end{cases}$$

admits a solution x.

Fiedler and Ptàk (1966b) prove the following result:

• The matrix A, of order (m,n), is an $\mathcal{S}_0$-matrix if and only if for every vector $p \geqq [0]$, $p \in \mathbb{R}^m$, at least one component of $p^\top A$ is nonnegative.

The classes $\mathcal{S}$ and $\mathcal{S}_0$ are obviously related by the inclusion $\mathcal{S} \subset \mathcal{S}_0$, but they are also related by the *Ville theorem of the alternative* (see, e. g., Cottle, Pang and Stone (2009), Gale (1960), Mangasarian (1969)):

• A be of order (m,n); then the system

$$Ax > [0]\,, \quad x > [0]$$

admits a solution if and only if the system

$$y^\top A \leqq [0]\,, \quad y \geq [0]$$

admits no solution.

In terms of the classes $\mathcal{S}$ and $\mathcal{S}_0$, the Ville theorem of the alternative can therefore be described by the following equivalence

$$(A \in \mathcal{S}) \iff ((-A^\top) \notin \mathcal{S}_0).$$

Moreover (Fiedler and Ptàk (1966b)): if $A \in \mathcal{P}$, then $A \in \mathcal{S}$; if $A \in \mathcal{P}_0$, then $A \in \mathcal{S}_0$.

Definition 10. A matrix A of order (m,n) is an $\mathcal{S}_1$-*matrix* or an *irreducibly* $\mathcal{S}_0$-*matrix,* if A belongs to $\mathcal{S}_0$ and either $n = 1$ (i. e. A has only one column) or $n > 1$ and no matrix obtained from A by omitting at least one column belongs to $\mathcal{S}_0$.

Fiedler and Ptàk (1966b) denote by M the above class of matrices. We have adopted the notation $\mathcal{S}_1$, in order to avoid confusions with the class of $\mathcal{M}$-matrices. Other characterizations of the $\mathcal{S}_1$-class of matrices are contained in the next theorem.

Theorem 13. Let A be a matrix of order (m,n). Then the following conditions are equivalent.

i) $A \in \mathcal{S}_1$.

ii) $A \in \mathcal{S}_0$ and the system

$$\begin{cases} Ax \geqq [0] \\ x \geq [0] \end{cases}$$

admits only solutions $x > [0]$.

iii) $A \in \mathcal{S}_0$ and for every $x \neq [0]$, solution of $Ax \geqq [0]$, it holds either $x > [0]$ or $x < Ax = [0]$.

iv) $A \in \mathcal{S}_0$ and, moreover, A verifies one of the following equivalent conditions.

iv.a) A admits a left-side generalized inverse $A^+ > [0]$, i. e. $A^+A = I$.

iv.b) For any vector $y \geq [0]$ there exists a solution $p > [0]$ of the system $p^\top A = y^\top$.

iv.c) $rk(A) = n$, and for every vector x such that $Ax \geq [0]$ it holds $x > [0]$.

iv.d) $rk(A) = n - 1$, and there exist vectors $p > [0]$ and $x > [0]$ such that $p^\top A = [0]$, $Ax = [0]$.

$iv.e$) If $Ax = [0]$ for $x \neq [0]$, then it holds either $Ax = [0] < x$ or $Ax = [0] > x$.

Other relevant results, proved by Fiedler and Ptàk (1966b), concerning the $\mathcal{S}_1$-class of matrices, are contained in the following theorem.

Theorem 14.

1) Let A be of order (m, n); if $A \in \mathcal{S}_1$, then $m \geqq n$.

2) Let A be a *square* matrix of order n; then, the following two conditions are equivalent:

a) $\det(A) \neq 0$ and $A \in \mathcal{S}_1$.

b) $A^{-1} > [0]$.

3) Let A be a *square* matrix of order $n > 1$; then, the following three conditions are equivalent:

α) Both A and $-A$ belong to $\mathcal{S}_1$.

β) $A \in \mathcal{S}_1$ and A is singular.

γ) A is singular and $adj(A)$ is either positive or negative.

On the grounds of Theorem 14, point 2), it turns out that if the pair (A, B), A and B both square of order n, and $A \geqq [0]$, $B \geqq [0]$, describe a *joint production model,* of the type considered by P. Sraffa, then $(B - A)^{-1} > [0]$, i. e. the model is *all-engaging,* in the terminology of B. Schefold (1978, 1989), if and only if $\det(B - A) \neq 0$ and $(B - A) \in \mathcal{S}_1$. This result is almost never considered in the economic literature. See also Giorgi and Magnani (1978), Giorgi (2014), Giorgi and Zuccotti (2014). As previously pointed out, all-engaging models are important in the theory of linear economic models with joint production, as they retain many relevant properties of single production models. See Schefold (1978).

6. Some Economic Applications

The class of $\mathcal{M}$-matrices and the other classes of matrices considered in the previous section have found several applications in a variety of fields: numerical analysis, linear complementarity problems, differential and difference equations, stochastic processes, economic models, problems of linear algebra, geometry, mathematical physics, etc. Economic applications of the class of $\mathcal{M}$-matrices are well known. See, e. g., Berman and Plemmons (1994), Giorgi and Magnani (1978), Kemp and Kimura (1978), Murata (1977), Nikaido (1968, 1970), Pasinetti (1977), Takayama (1985), Woods (1978). Here we shall be concerned only with some economic applications of the $\mathcal{S}$-class, the $\mathcal{S}_0$-class and the $\mathcal{S}_1$-class. Some insights have been already given in the previous section.

We consider a general linear economic model with joint production, described by two nonnegative matrices:

- An inputs matrix $A \geqq [0]$, of order (m, n);
- An outputs matrix $B \geqq [0]$, of order (m, n).

Usually, due to the economic meaning of A and B, every column of A and B is required to be semipositive:

$$A^j \geq [0], \ B^j \geq [0], \quad \forall j = 1, ..., n.$$

Also every row of B is required to be semipositive:

$$B_i \geq [0], \quad \forall i = 1, ..., m.$$

This last assumption means that every good can be produced by some process (the model is "complete"). See, e. g., Giorgi and Magnani (1978), Kemeny, Morgenstern and Thompson (1956). The nonnegative column vector $x \in \mathbb{R}^n$ is the *activity vector,* therefore the quantities Bx and Ax describe, respectively, the gross productions and the inter-industry consumes. The row vector $p^\top \in \mathbb{R}^m$ (usually $p \geq [0]$ or also $p > [0]$) is the *price vector.* The vector $y = (B-A)x$ describes the *net productions,* obtained at the activity levels vector x, and the vector $v^\top = p^\top(B-A)$ describes the unitary net values, i. e. the values, at the price vector p, referred to the activity vector $x = e$, with $e = [1, 1, ..., 1]^\top$.

The model is *productive* if there exists an activity vector $x \geq [0]$ such that y is positive:

$$y = (B-A)x > [0], \ x \geq [0] \ \text{(equivalently: } x > [0]\text{)}.$$

The productivity of the model here considered is therefore equivalent to the property

$$(B-A) \in \mathcal{S}.$$

The model is *profitable* if there exists a price vector $p \geq [0]$ such that $v > [0]$:

$$v^\top = p^\top(B-A) > [0], \quad p \geq [0] \ \text{(equivalently: } p > [0]\text{)}.$$

The profitability is therefore equivalent to the property

$$(B-A) \in \mathcal{S}^\top$$

which means $(B-A)^\top \in \mathcal{S}$.

Unlike for the class of $\mathcal{M}$-matrices ("closed" under transposition), we have to remark that the two properties of productivity and profitability are compatible, but not *independent* properties. In other words, the classes $\mathcal{S}$ and $\mathcal{S}^\top$ are not disjoint, but have only a partial overlapping. Obviously, if A and B are *square,* of the same order n, and $(B-A) \in \mathcal{Z}$, then productivity is just equivalent to profitability. In the general case we can formulate the following "test" of productivity and profitability for a general linear production model, not necessarily square, descriveb by the pair (A, B).

Theorem 15. Let A and B be, respectively, the inputs and the outputs matrix of an economic linear model involving m commodities and n processes. Then:

i) The model (A, B) is productive if and only if, for any price vector $p \geq [0]$, there exists an activity (in general varying with the choice of p) such that the corresponding net value is positive:

$$p \geq [0] \Longrightarrow \exists j : p^\top(B-A)^j > 0.$$

ii) The model (A, B) is profitable if and only if, for any activity vector $x \geq [0]$, there exists a commodity (in general varying with the choice of x) such that the corresponding net production is positive:

$$x \geq [0] \Longrightarrow \exists i : (B - A)_i x > 0.$$

Proof. Thanks to the theorem of the alternative of Ville (see Section 5), $(B - A) \in \mathcal{S}$ if and only if $\left[-(B - A)^\top\right] \notin \mathcal{S}_0$. This means that the system

$$\begin{cases} \left[-(B - A)^\top\right] p \geqq [0] \\ p \geq [0] \end{cases}$$

i. e. the system

$$\begin{cases} p^\top (B - A) \leqq [0] \\ p \geq [0] \end{cases}$$

has a solution. Therefore *i)* is proved. In a symmetric way, $(B - A)$ is profitable if and only if $(B - A)^\top \in \mathcal{S}$, i. e., thanks to the same theorem of the alternative, if and only if $[-(B - A] \notin \mathcal{S}_0$. This means that the system

$$\begin{cases} [-(B - A)]\, x \geqq [0] \\ x \geq [0], \end{cases}$$

i. e. the system

$$\begin{cases} (B - A)x \leqq [0] \\ x \geq [0] \end{cases}$$

has no solution. Therefore *ii)* is proved. □

Obviously, the practical relevance of the above tests relies on the possibility to detect non productive models and non profitable models, rather than productive models or profitable models.

If $(B - A) \in \mathcal{S}_1$, $(B - A)$ *not square,* the number of processes is always less than the number of commodities $(n < m)$, thanks to Property 3.8 of Fiedler and Ptåk (1966b). Moreover, if the columns of $(B - A)$ are linearly independent, then the pair (A, B) is profitable and *quasi-productive,* i. e. the system

$$\begin{cases} (B - A)x \geq [0] \\ x \geqq [0] \end{cases}$$

has a solution (see Giorgi and Magnani (1978)).

If A and B are square, of the same order n, $(B - A)$ is nonsingular and $(B - A) \in \mathcal{S}_1$ (i. e. the model is "all-engaging": $(B - A)^{-1} > [0]$), then, again productivity and profitability are equivalent properties. If $(B - A) \in \mathcal{P}$, and hence also $(B - A)^\top \in \mathcal{P}$, then the linear production model described by the pair (A, B) is both productive and profitable.

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