ISSN: 2281-1346

**DEM Working Paper Series**

# On Second-Order Optimality Conditions in Smooth Nonlinear Programming Problems

Giorgio Giorgi

(University of Pavia)

**# 171 (03-19)**

Via San Felice, 5
I-27100 Pavia

economiaweb.unipv.it

# On Second-Order Optimality Conditions in Smooth Nonlinear Programming Problems

Giorgio Giorgi*

Abstract The aim of the present paper is to give an overview of the basic facts concerning second-order optimality conditions (necessary and sufficient) for a nonlinear programming problem under twice continuous differentiability assumptions. Some remarks and some new propositions are pointed out.

Key words and phrases Nonlinear programming, second-order optimality conditions, sensitivity, constraint qualifications.

Mathematics Subject Classification (2010): 90C29, 90C30, 49K27.

## 1. Introduction

The quest for second-order optimality conditions for unconstrained optimization problems started a long time ago, together with the basic tools of Calculus and Mathematical Analysis. For "classical" constrained optimization problems, i. e. mathematical programming problems with *equality constraints only,* one of the first basic reference work is the book of Hancock (1960), originally published in 1917. Other standard works on this subject are Caratheodory (1989), Frisch (1966) and Burger (1955). For "modern" constrained optimization problems, i. e. mathematical programming problems with inequality or both inequality and equality constraints, one of the first treatments of second-order optimality conditions is contained in the Master Thesis of W. Karush (1939). Other "historical" papers on these topics are Pennisi (1953), Pallu de la Barrière (1963) and McCormick (1967).

The literature on second-order optimality conditions (necessary and sufficient) for nonlinear programming problems is indeed abundant. Quite recently several papers dealing with "sophisticated" tools have appeared, some of them taking into considerations second-order tangent sets and second-order tangent cones, in order to express second-order optimality conditions also for nonsmooth optimization problems. See, e. g., Cambini, Martein and Vlach (1999), Cambini and Martein (2003), Bonnans, Cominetti and Shapiro (1999), Kawasaki (1988), Cominetti (1990), Penot (1998, 2000), Jiménez and Novo (2004) and the survey paper of Giorgi, Jiménez and Novo (2010).

* Department of Economics and Management, Via S. Felice, 5 - 27100 Pavia, (Italy). E-mail: giorgio.giorgi@unipv.it

Here we treat only the case of "smooth" functions, i. e. the case of twice continuously differentiable functions, being this property more easy to check than the weaker "twice differentiability". We remark however that several results of the present paper hold under twice differentiability. There are various definitions of this concept (see, e. g., Rockafellar and Wets (2009)); we accept the following one. See, e. g., Pagani and Salsa (1990), Roux (1984), Hiriart-Urruty (2008).

Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable (in the Fréchet sense) on the open set $X$. We say that $f$ is *twice differentiable at* $\bar{x} \in X$ when $\nabla f : X \longrightarrow \mathbb{R}^n$ is differentiable at $\bar{x}$. The Jacobian matrix of $\nabla f$ is called *Hessian matrix* of $f$ at $\bar{x}$ and denoted $Hf(\bar{x})$ or $\nabla^2 f(\bar{x})$. This matrix is *symmetric,* of order $n$, and we have the Taylor-Peano-Young formula

$$f(\bar{x}+h) = f(\bar{x}) + \nabla f(\bar{x})h + \frac{1}{2}h^\top \nabla^2 f(\bar{x})h + o(\|h\|^2),$$

for $h \longrightarrow 0$.

Rockafellar and Wets (2009) give various definitions of twice differentiability and one of them coincides with the above definition. They speak of "twice differentiability in the classical sense". Curiously, on the same page these authors add the following sentence: "of course, even for Hessian matrices in the classical sense symmetry can't be taken for granted", which is a flaw. See, e. g., Pagani and Salsa (1990), Roux (1984).

In spite of numerous books and papers where second-order optimality conditions are developed, there are still some questions worthy to be remarked; moreover, in some textbooks there are errors which may generate in the readers some false convinctions.

Aim of the present paper is to give an overview of the basic facts concerning second-order optimality conditions for a nonlinear programming problem, under twice continuous differentiability assumptions, together with some remarks and some new propositions.

The paper is organized as follows. In the Introduction the basic definitions, tools and properties concerning optimality conditions for a mathematical programming problem are surveyed. Section 2 is concerned with second-order necessary optimality conditions. Section 3 takes into consideration some other approaches to second-order necessary conditions under various constraint qualifications and assumptions. Section 4 is concerned with sufficient second-order optimality conditions. Section 5 considers some other approaches in obtaining sufficient second-order optimality conditions. The final Section 6, based on results of Fiacco (1976, 1980, 1983) and Fiacco and Kyparisis (1985), points out some applications of second-order optimality conditions to sensitivity analysis for a parametric nonlinear programming problem.

Second-order optimality conditions for vector optimization problems are treated, e. g., in Aghezzaf (1999), Aghezzaf and Hachimi (1999), Bigi (2006), Bigi and Castellani (2000, 2004), Bolintineanu and El Maghri (1998), R. Cambini (1998), Cambini and Martein (2003), Giorgi, Jiménez and Novo (2010), Gutierrez, Jiménez and Novo (2009, 2010), Jiménez and Novo (2002, 2003, 2004), Hachimi and Aghezzaf (2007), Kim and Tuyen (2018), Maeda (2004), Maciel, Santos and Sottosanto (2011, 2012), Ning, Song and Zhang (2012), Ritzvi and Nasser (2006), Wang (1991).

**Definition 1.** A sequence $\left\{x^k\right\} \subset \mathbb{R}^n \diagdown \{x^0\}$, with $x^k \longrightarrow x^0$ is called *tangentially convergent* in the direction $y \in \mathbb{R}^n$ to the point $x^0$ if

$$\lim_{k \longrightarrow +\infty} \frac{x^k - x^0}{\|x^k - x^0\|} = y$$

and we write $x^k \xrightarrow{y} x^0$.

Obviously any convergent sequence $x^k \longrightarrow x^0$ (with $x^k \neq x^0$ for all $k$) contains at least a tangentially convergent subsequence. The set of all directions $y$ for which there exists a feasible sequence $\left\{x^k\right\} \subset S$, with $S \subset \mathbb{R}^n$, tangentially convergent to $x^0 \in S$, form a cone which is a local cone approximation at $x^0$ of the set $S \subset \mathbb{R}^n$.

**Definition 2.** Let $S \subset \mathbb{R}^n$ and $x^0 \in S$; the cone

$$T(S; x^0) = \left\{\lambda y \in \mathbb{R}^n : \exists \left\{x^k\right\} \subset S,\ x^k \xrightarrow{y} x^0,\ \ \lambda \geqq 0\right\}$$

is called *Bouligand tangent cone or contingent cone* to the set $S$ at the point $x^0$. If $x^0$ is an isolated point of $S$, we set $T(S; x^0) = \{0\}$.

Other equivalent characterizations of $T(S; x^0)$ are the following ones (see, e. g., Aubin and Frankowska (1990), Bazaraa and Shetty (1976), Bazaraa, Goode and Nashed (1974), Giorgi and Guerraggio (1992, 2002)):

$$T(S; x^0) = \left\{ \begin{array}{l} y \in \mathbb{R}^n : \exists \left\{x^k\right\} \subset S,\ \lim\limits_{k \longrightarrow +\infty} x^k = x^0,\ \exists \{\lambda_k\} \subset \mathbb{R}_+ \text{ such that} \\ y = \lim\limits_{k \longrightarrow +\infty} \lambda_k (x^k - x^0); \end{array} \right\}$$

$$T(S; x^0) = \left\{y \in \mathbb{R}^n : \exists \left\{y^k\right\} \longrightarrow y,\ \exists \{t_k\} \longrightarrow 0, \text{ such that } x^0 + t_k y^k \in S\right\};$$

$$T(S; x^0) = \left\{y \in \mathbb{R}^n : \forall N(y),\ \forall \lambda > 0,\ \exists t \in (0, \lambda),\ \exists \bar{y} \in N(y) \text{ such that } x^0 + t\bar{y} \in S\right\}.$$

Note that $T(S; x^0)$ is indeed a cone, with $0 \in T(S; x^0)$. We note also that:

*i)* $T(S; x^0)$ depends only from the structure of $S$ in a neighborhood of $x^0$, that is

$$T(S; x^0) = T(S \cap U(x^0); x^0),$$

where $U(x^0)$ is any neighborhood of $x^0$ (the notion of "Bouligand tangent cone" is therefore an "infinitesimal notion").

*ii)* If $x^0 \in int(S)$, then $T(S; x^0) = \mathbb{R}^n$.

*iii)* $T(S; x^0) = T(\bar{S}; x^0)$, where $\bar{S} = cl(S)$.

*iv)* $T(S_1; x^0) \subset T(S_2; x^0)$, if $x^0 \in S_1 \subset S_2$.

*v)* $\quad T\left(\cup_{i=1}^k S_i; x^0\right) = \cup_{i=1}^k T(S_i; x^0)$. (There are no similar relations for the intersection operation).

*vi)* $\quad T(S; x^0)$ is always a *closed* cone, but not necessarily convex.

**Theorem 1.** Let $x^0 \in S \subset \mathbb{R}^n$ and let $S$ be a convex set; then it holds

$$T(S; x^0) = cl(cone(S - x^0)).$$

Here $cone(S)$ is the convex cone generated by $S$, i. e.

$$cone(S) = \left\{ \sum_{i=1}^{k} \lambda_i x^i : x^i \in S,\ \lambda_i \geqq 0,\ \sum_{i=1}^{k} \lambda_i > 0,\ k \in \mathbb{N} \right\}.$$

This cone is the intersection of all convex cones that contain $S$.

If $S \subset \mathbb{R}^n$ is a nonempty set, we denote by $S^*$ the (negative) *polar cone* of $S$, given by

$$S^* = \{ y \in \mathbb{R}^n : yx \leqq 0,\ \forall x \in S \}.$$

If $S$ is empty, then $S^*$ is interpreted as the whole space $\mathbb{R}^n$. Note that $S^*$ is a closed convex cone; moreover, it holds

$$S^* = (cl(S))^*.$$

One may also define the *bipolar cone* $S^{**}$ of $S$ by the relation

$$S^{**} = (S^*)^*.$$

It results $S \subset S^{**}$, but if $S$ is a nonempty *convex cone* in $\mathbb{R}^n$, it holds

$$S^{**} = cl(S)$$

and if the convex cone $S$ is also closed, then obviously it holds

$$S = S^{**}.$$

If $S \subset \mathbb{R}^n$ is a nonempty set, we denote by $conv(S)$ the *convex hull* of $S$.

We consider the following nonlinear programming problem, with both equality and inequality constraints.

$$(P): \quad \begin{cases} \min f(x) \\ \text{subject to:} \quad g_i(x) \leqq 0, \quad i \in M; \\ \qquad\qquad\qquad h_j(x) = 0, \quad j \in P, \end{cases}$$

where $M = \{1, ..., m\}$, $P = \{1, ..., p < n\}$, $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $g_i : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$; $h_j : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, ..., p < n$.

Every function $f$, $g_i$, $h_j$ is assumed to be twice continuously differentiable on the open set $X$.

With reference to $(P)$ we denote by $K$ its *feasible set:*

$$K = \{x \in \mathbb{R}^n : g_i(x) \leqq 0, \ \forall i \in M; \ h_j(x) = 0, \ \forall j \in P\}.$$

We assume $K \neq \varnothing$.

Let $x^0 \in K$; the index set of *active inequality constraints* at $x^0$ for $(P)$ is denoted by $I(x^0)$, i. e.

$$I(x^0) = \left\{i \in M : g_i(x^0) = 0\right\}.$$

We recall some basic definitions regarding a solution point $x^0$ for $(P)$ (see, e. g., Giorgi and Zuccotti (2008), Still and Streng (1996)).

**Definition 3.** Let $x^0 \in K$; then the point $x^0$ is:

*a)* A *local minimum point* for $(P)$ if there is a neighborhood $U(x^0)$ of $x^0$ such that

$$f(x) \geqq f(x^0), \quad \forall x \in K \cap U(x^0).$$

*b)* A *strict local minimum point* for $(P)$ if there is a neighborhood $U(x^0)$ of $x^0$ such that

$$f(x) > f(x^0), \quad \forall x \neq x^0, \quad x \in K \cap U(x^0).$$

*c)* A *strict local minimum point of order* $p$ for $(P)$, if there are a neighborhood $U(x^0)$ and a constant $k > 0$ such that, for $p \in \mathbb{N}$,

$$f(x) \geqq f(x^0) + k \left\|x - x^0\right\|^p, \ \forall x \in K \cap U(x^0).$$

If $p = 1$, the point $x^0$ is more commonly called a *strong local minimum point* for $(P)$ or also a *sharp local minimum point* (see, e. g., Shapiro and Al-Khayyal (1993); see also Ferris and Mangasarian (1992), Burke and Ferris (1993), Ward (1994) who give a more general definition than the one reported in the present paper). If $p = 2$, some authors speak of "quadratic growth condition".

*d)* An *isolated local minimum point* for $(P)$, if there is a neighborhood $U(x^0)$ such that $x^0$ is the *unique* local minimizer in $U(x^0) \cap K$.

We observe that if $x^0$ is a strict local minimum point of order $p$, then it is also a strict local minimum point of order $r$ for all $r > p$.

Moreover, it is clear that any strict local minimizer of order $p$ is a strict local minimizer. However, not every strict local minimizer is a strict local minimizer of order $p$ for some $p$. For example (see Ward (1994)), define $f : [0, +\infty) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} f(x) &= x^{\frac{1}{x}}, \quad \text{for } x > 0 \\ f(0) &= 0 \end{aligned}$$

and let $K = [0, +\infty)$. Then $x^0 = 0$ is a strict local minimizer that is not a strict local minimizer of order $p$ for any $p$.

The concept of strict local minimum point of order $p$ was studied by Cromme (1978) in relation to iterative numerical methods, but was previously introduced by Hestenes (1966).

While strict local minimizers are not always isolated, it is true that all isolated local minimizers are strict. For example, if we consider $f(x) = 1$, if $x \neq 0$ and $f(0) = 0$, we have that 0 is a strict minimizer, but not an isolated local minimizer. A more elaborate function is considered by Fletcher (1987):

$$f(x) = x^2(1 + x^2 + \sin(\frac{1}{x})), \quad x \neq 0, \ f(0) = 0.$$

Here $x^0 = 0$ is a strict local minimizer, but not isolated. For *unconstrained* minimization problems, the usual second-order sufficient conditions:

$$\nabla f(x^0) = 0, \quad y^\top \nabla^2 f(x^0) y > 0, \quad \forall y \in \mathbb{R}^n, \ y \neq 0,$$

are sufficient for $x^0$ to be a strict and isolated local minimizer (see, e. g., Fletcher (1987)). This is *not* the case for *constrained* minimization problems, for example for $(P)$. See further in the present paper.

With reference to $(P)$ we now recall the well-known (first-order) *Karush-Kuhn-Tucker conditions.*

**Theorem 2.** Suppose that $x^0 \in K$ is a local minimum point of $(P)$ and that an appropriate *constraint qualification* holds at $x^0$. Define the *Kuhn-Tucker-Lagrange function* associated with $(P)$ as follows:

$$\mathcal{L}(x, u, w) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x).$$

Then the Karush-Kuhn-Tucker conditions (KKT) hold at $x^0$, i. e. there exist multiplier vectors $u$ and $w$ such that

$$(KKT): \qquad \begin{cases} \nabla_x \mathcal{L}(x^0, u, w) = 0, \\ u_i g_i(x^0) = 0, \ i = 1, ..., m, \\ u_i \geqq 0, \ i = 1, ..., m. \end{cases}$$

There are many constraint qualifications which suffice for Theorem 2 to hold; for a quite recent up-to-date survey, under differentiability assumptions, see Giorgi (2018). Here we recall some constraint qualifications that will be used in the present paper. Other constraint qualifications will be introduced subsequently.

Let $x^0 \in K$.

$(a)$ The *Mangasarian-Fromovitz Constraint Qualification* (MFCQ) holds at $x^0$ if

$(i)$ The vectors $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent;

$(ii)$ There is $z \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(x^0)z &< 0,\ i \in I(x^0), \\ \nabla h_j(x^0)z &= 0,\ j = 1, ..., p. \end{aligned}$$

Applying a theorem of the alternative (see, e. g., Mangasarian (1969)), the equivalent dual form of (MFCQ) states that

- zero is the unique solution of the relation

$$\sum_{i \in I(x^0)} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} w_j \nabla h_j(x^0) = 0,$$

$$u_i \geqq 0, \quad \forall i \in I(x^0).$$

$(b)$ The *Linear Independence Constraint Qualification* (LICQ) holds at $x^0$ if the vectors

$$\nabla g_i(x^0),\ i \in I(x^0);\ \nabla h_j(x^0),\ j = 1, ..., p,$$

are linearly independent.

$(c)$ The *Strict Mangasarian-Fromovitz Constraint Qualification* (SMFCQ) holds at $x^0$ if, denoting by $I^+(x^0, u)$ the set of *strictly active* inequality constraints at $x^0$, i. e.

$$I^+(x^0, u) = \left\{ i \in I(x^0) \text{ and there is } (u, w) \text{ satisfying (KKT) with } u_i > 0 \right\},$$

$i)$ The gradients $\nabla g_i(x^0)$, $i \in I^+(x^0, w)$; $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent.

$ii)$ The system

$$\begin{aligned} \nabla g_i(x^0)z &< 0,\ i \in I(x^0) \diagdown I^+(x^0, u) \\ \nabla g_i(x^0)z &= 0,\ i \in I^+(x^0, u) \\ \nabla h_j(x^0)z &= 0,\ j = 1, ..., p, \end{aligned}$$

has a solution $z \in \mathbb{R}^n$.

(in Giorgi and Zuccotti (2008) there is a misprint: it is required the linear independence of the vectors $\nabla g_i(x^0)$, $i \in I(x^0)$; $\nabla h_j(x^0)$, $j = 1, ..., p$, instead of $\nabla g_i(x^0)$, $i \in I^+(x^0, w)$; $\nabla h_j(x^0)$, $j = 1, ..., p$).

$(d)$ The *Constant Rank Condition* (CRC) holds at $x^0$ if for any subset $L \subset I(x^0)$ of active constraints and $N \subset P$ of equality constraints, the family

$$\{\nabla g_i(x),\ i \in L,\ \nabla h_j(x),\ j \in N\}$$

remains of constant rank near the point $x^0$.

Conditions $(a)$-$(d)$ are well-known on the literature. In particular, condition $(c)$ was introduced by Kyparisis (1985) who has shown that this condition is both necessary and sufficient to have uniqueness of multiplier vectors in (KKT) conditions. In other words, we have the following implications:

$$(LICQ) \Longrightarrow (SMFCQ) \iff \{\text{uniqueness of (KKT) multipliers}\} \Longrightarrow (MFCQ).$$

Note that, however, the (SMFCQ) condition is not properly a constraint qualification, as it involves the multiplier vectors in its definition. Usually, these multiplier vectors depend also from the objective function $f$; see, e. g., Wacksmuth (2013). Perhaps it is better to call (SMFCQ) a "regularity condition".

The (CRC) condition was introduced by Janin (1984) and was used by Andreani, Echagüe and Schuverdt (2010) in obtaining second-order optimality conditions. Clearly, (LICQ) implies (CRC). Linearity of the constraints also implies (CRC). The (CRC) condition is neither weaker nor stronger than (MFCQ). (CRC) is indeed a first-order constraint qualification, in the sense that if $x^0 \in K$ is a local minimizer of $(P)$ and (CRC) holds, then (KKT) conditions hold. Note also that unlike (MFCQ), if (CRC) holds at $x^0 \in K$, it will continue to hold if any of the equality constraints $h_j(x) = 0$ were to be replaced by the two inequalities $h_j(x) \leqq 0$ and $-h_j(x) \leqq 0$. (MFCQ) and (CRC) conditions are related, however, in the following sense: it can be shown that under (CRC) there exists an alternative representation of the feasible set for which (MFCQ) holds; see Lu (2011, 2012), Giorgi (2018).

Another well-known (first-order) necessary optimality condition for $(P)$ is expressed by the following result, originally due to Fritz John (1948).

**Theorem 3.** Suppose that $x^0 \in K$ is a local minimum of $(P)$ and define the *Fritz John-Lagrange function* associated to $(P)$ as follows:

$$\mathcal{L}_1(x, u_0, u, w) = u_0 f(x) + \sum_{i=1}^{n} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x).$$

Then the Fritz John conditions (FJ) hold at $x^0$, i. e. there exist multiplier vectors $(u_0, u)$ and $w$, *not simultaneously zero,* such that

$$(FJ): \qquad \begin{cases} \nabla_x \mathcal{L}_1(x^0, u_0, u, w) = 0, \\ u_i g_i(x^0) = 0, \quad i = 1, ..., m, \\ u_0 \geqq 0, \; u_i \geqq 0, \; i = 1, ..., m. \end{cases}$$

**Remark 1.** If we assume that the gradients $\nabla h_j(x^0)$, $j = 1, ..., p$, are linearly independent, it is possible to reformulate the (FJ) conditions in a more stringent form (see Still and Streng (1996)): there exist multiplier vectors $(u_0, u) \geqq 0$, *not identically zero,* and $w$, such that the (FJ) conditions hold.

The set of multipliers $(u_0, u, w)$ satisfying at $x^0 \in K$ the (FJ) conditions (set of *Fritz John multipliers or generalized Lagrange multipliers*) is denoted by $\Lambda_0(x^0)$. The set of multipliers $(u, w)$ satisfying at $x^0 \in K$ the (KKT) conditions (set of *Karush-Kuhn-Tucker multipliers*) is denoted by $\Lambda(x^0)$. In other words $\Lambda(x^0) = \left\{(u,w) \in \mathbb{R}^m_+ \times \mathbb{R}^p : (1,u,w) \in \Lambda_0(x^0)\right\}$. We have the following propositions (see Bonnans and Shapiro (2000), Gauvin (1977), Kyparisis (1985)).

**Theorem 4.** Let $x^0 \in K$ be a local solution of $(P)$. Then $\Lambda_0(x^0)$ is nonempty and the following conditions are equivalent:

1) The (MFCQ) condition holds at $x^0$.

2) The set $\Lambda_0(x^0)$, where $u_0 = 0$, is empty.

3) The set $\Lambda_0(x^0) = \Lambda(x^0)$ is convex and compact.

**Theorem 5.** Let $x^0 \in K$ be a local solution of $(P)$. Then the following conditions are equivalent:

*i)* The (SMFCQ) condition holds at $x^0$.

*ii)* The set $\Lambda(x^0)$ is a singleton.

## 2. Second-Order Necessary Optimality Conditions

With reference to problem $(P)$ let us define the so-called *critical cone or cone of critical directions* $Z(x^0)$, where $x^0 \in K$ :

$$Z(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla g_i(x^0)z = 0, \quad i \in I^+(x^0 u), \\ \nabla g_i(x^0)z \leqq 0, \quad i \in I(x^0) \diagdown I^+(x^0, u), \\ \nabla h_j(x^0)z = 0, \quad j = 1, ..., p. \end{array} \right\}$$

The next theorem states the classical second-order necessary conditions (SONC) for local optimality of problem $(P)$. These conditions are essentially due to McCormick (1967, 1976, 1983). See also Fiacco and McCormick (1968a).

**Theorem 6.** Suppose tnat $x^0 \in K$ is a local solution of $(P)$ and that the (LICQ) condition holds at $x^0$. Then, the (KKT) conditions hold at $x^0$ with associated unique multiplier vectors $u$ and $w$, and the additional Second-Order Necessary Conditions (SONC) hold at $x^0$ :

$$z^\top \nabla^2_x \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z(x^0).$$

In the literature another description of the cone $Z(x^0)$ often appears; it is the cone (again called "critical cone"):

$$C(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla f(x^0)z = 0, \\ \nabla g_i(x^0)z \leqq 0, \quad i \in I(x^0), \\ \nabla h_j(x^0)z = 0, \quad j = 1, ..., p. \end{array} \right\}$$

It can be proved that, under the validity of the (KKT) conditions at $x^0 \in K$, the two cones $C(x^0)$ and $Z(x^0)$ coincide (in Han and Mangasarian (1979) there are some minor inaccuracies).

**Theorem 7.** Let $x^0 \in K$ verify the (KKT) conditions. Then

$$C(x^0) = Z(x^0).$$

**Proof.** We first show that $C(x^0) \subset Z(x^0)$. Let $z \in C(x^0)$; clearly we only need to show that for $i \in I^+(x^0, u)$ we have $\nabla g_i(x^0)z = 0$. By (KKT) we have that

$$\nabla f(x^0)z + \sum_{i \in I(x^0)} u_i \nabla g_i(x^0)z + \sum_{j=1}^{p} w_j \nabla h_j(x^0)z = 0.$$

Because $\nabla h_j(x^0)z = 0$, $j = 1, ..., p$ and $u_i = 0$ for $i \in I(x^0) \diagdown I^+(x^0 u)$, we have

$$\nabla f(x^0)z + \sum_{i \in I^+(x^0,u)} u_i \nabla g_i(x^0)z = 0.$$

Because $\nabla f(x^0)z = 0$, and every $u_i > 0$ for all $i \in I^+(x^0, u)$, we have

$$\nabla g_i(x^0)z = 0, \quad i \in I^+(x^0, u).$$

Now we prove that $Z(x^0) \subset C(x^0)$. Let $z$ be any point in $Z(x^0)$. It suffices to show that $\nabla f(x^0)z = 0$. As before we have

$$\nabla f(x^0)z + \sum_{i \in I(x^0)} u_i \nabla g_i(x^0)z + \sum_{j=1}^{p} w_j \nabla h_j(x^0)z = 0.$$

Clearly $\nabla f(x^0)z = 0$ because all the other terms are zero. □

The (SONC) expressed in Theorem 6 are called by some authors "strong second-order necessary optimality conditions" for $(P)$. Note (see Kyparisis (1985)) that the (LICQ) can be substituted by the weaker (SMFCQ), but *not* by the (MFCQ): in other words, (MFCQ) is *not a second-order constraint qualification,* which assures the validity of Theorem 6. This has been remarked by Arutyunov (1991), Anitescu (2000), Baccari (2004).

**Remark 2.** For an algorithmic point of view, another set taken into consideration to formulate (SONC) is the following one.

$$Z_1(x^0) = \left\{ z \in \mathbb{R}^n : \nabla g_i(x^0)z = 0, \quad i \in I(x^0); \; \nabla h_j(x^0)z = 0, \quad j = 1, ..., p, \right\}$$

called *critical subspace* or also *weak critical cone* at $x^0 \in K$. See, e. g., McCormick (1967, 1976, 1983), Fiacco and McCormick (1968a). Being $Z_1(x^0) \subset Z(x^0)$, the following result is obvious.

**Corollary 1.** Assume as in Theorem 6. Then, an additional (SONC) for $x^0$ is:

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z_1(x^0).$$

The (SONC) of Corollary 1 are sometimes called *weak second-order necessary optimality conditions* for $(P)$. However, we prefer to use this denomination for the second-order conditions with *non-fixed multipliers* $(u_0, u, w)$, introduced by Hettich and Jongen (1977) and

Ben-Tal (1980). See further in the present Section. We propose for the condition of Corollary 1 the denomination "classical strong second-order necessary optimality condition" or (following Ruszczynski (2006)) "semi-strong second-order necessary optimality condition". We call the reader's attention to the fact that there is not uniformity of denominations in the literature about second-order optimality conditions. See, e. g., Guo, Lin and Ye (2013) for other classifications.

It must be noted that the (less sharp) conditions of Corollary 1 can be tested by matrix techniques: indeed we have to test the sign of a quadratic form subject to a system of homogeneous linear equations. It is well-known that there are necessary and sufficient algorithms to check the sign of this type of constrained quadratic forms. See, e. g., Chabrillac and Crouzeix (1984), Debreu (1952), Giorgi (2017). See also McCormick (1983, page 222) for other considerations. If in the (KKT) conditions we have $u_i > 0$, $\forall i \in I(x^0)$, i. e. the *Strict Complementarity Slackness Conditions* hold at $x^0$, then obviously $Z(x^0) = Z_1(x^0)$.

**Remark 3.** With reference to problem $(P)$ it is *not* true the conjecture that if $x^0 \in K$ is a local minimizer, then the Hessian matrix $\nabla^2 f(x^0)$ is positive semidefinite. Consider, e. g., the following example, where $x \in \mathbb{R}^2$.

$$\begin{aligned} &\min \{-(x_1+1)^2 - (x_2)^2\} \\ &\text{subject to: } (x_1)^2 + (x_2)^2 \leqq 1. \end{aligned}$$

The optimal point is $x^0 = (1, 0)$, but $\nabla^2 f(x^0)$ is not positive semidefinite.

Another *false* conjecture is the following one: if $x^0 \in K$ is a local minimizer for $(P)$, then the Hessian matrix of the Lagrangian $\nabla_x \mathcal{L}(x^0 u, w)$ is positive semidefinite. Consider, e. g., the following example, where $x \in \mathbb{R}^2$.

$$\begin{aligned} &\min \{-(x_1)^2 + (x_2)^2\} \\ &\text{subject to:} \begin{cases} x_1 \leqq 1 \\ -x_2 \leqq 0. \end{cases} \end{aligned}$$

The optimal point is $x^0 = (1, 0)$ but $\nabla_x \mathcal{L}(x^0 u, w)$ is not positive semidefinite.

We have remarked that (MFCQ) does not assure the validity of Theorem 6. How is it possible, under (MFCQ) or *any first-order constraint qualification,* to obtain some second-order necessary optimality conditions for $(P)$? In order to give a first answer to this question we need a preliminary result.

Let us consider a minimization problem with an *abstract constraint* (or *set constraint*):

$$(P_1): \qquad \min_{x \in S} f(x)$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is twice continuously differentiable on an open set containing the set $S \subset \mathbb{R}^n$, on which no particular topological property is a priori made (it may be closed, neither closed nor open, etc). We recall (see, e. g., Bazaraa and Shetty (1976), Giorgi, Guerraggio and Thierfelder (2004), Gould and Tolle (1971)) that a basic first-order necessary optimality criterion for $(P_1)$ is the following one.

**Theorem 8.** Let $x^0 \in S$ be a local minimum point for $(P_1)$; then we have

$$\nabla f(x^0)y \geqq 0, \quad \forall y \in T(S; x^0).$$

A second-order necessary optimality condition for $(P_1)$ is given by the following result.

**Theorem 9.** Let $x^0 \in S$ be a local solution of $(P_1)$. Let $\nabla f(x^0) = 0$. Then it holds

$$y^\top \nabla^2 f(x^0)y \geqq 0, \quad \forall y \in T(S; x^0).$$

**Proof.** Let $y \neq 0$ be any vector of $T(S; x^0)$. Without loss of generality let $\|y\| = 1$; then it exists a feasible sequence $\{x^k\} \subset S$, with $x^k \xrightarrow{y} x^0$. Being $x^0$ a local minimum point of $(P_1)$, the quotients

$$\frac{f(x^k) - f(x^0)}{\|x^k - x^0\|^2} = \frac{\frac{1}{2}(x^k - x^0)^\top \nabla^2 f(x^0)(x^k - x^0) + o(\|x^k - x^0\|^2)}{\|x^k - x^0\|^2}$$

for $k \in \mathbb{N}$ large enough, are nonnegative and convergent to $\frac{1}{2}y^\top \nabla^2 f(x^0)y$. The thesis is therefore proved. □

**Remark 4.** If $x^0 \in int(S)$, surely it holds $\nabla f(x^0) = 0$ and being in this case $T(S; x^0) = \mathbb{R}^n$, Theorem 9 recovers the classical second-order necessary optimality conditions for an unconstrained optimization problem. It is possible to prove a weaker version of Theorem 9:

$$y \in T(S; x^0), \quad \nabla f(x^0)y = 0 \quad \Longrightarrow \quad y^\top \nabla^2 f(x^0)y \geqq 0,$$

if $S$ is a *convex polyhedral set.*

Let us now consider the problem with both inequality and equality constraints, i. e. problem $(P)$. Let $x^0$ be a local minimum point for $(P)$ and let some constraint qualification be satisfied at $x^0$ (for example the (MFCQ) or the Guignard-Gould-Tolle CQ, which is the most general CQ; see Gould and Tolle (1971), Giorgi (2018)). Then, at $x^0$ the (KKT) conditions hold. Let be

$$K_1 = \left\{x \in K : g_i(x) = 0, \ \forall i \in I^+(x^0, u)\right\},$$

where

$$I^+(x^0, u) = \left\{i \in I(x^0) : u_i > 0\right\} \subset I(x^0).$$

We recall that $\mathcal{L}(x, u, w)$ is the usual Lagrangian function for $(P)$ :

$$\mathcal{L}(x, u, w) = f(x) + ug(x) + wh(x), \ u \geqq 0.$$

Thanks to the complementarity slackness conditions we have

$$\mathcal{L}(x, u, w) = f(x), \quad \forall x \in K_1.$$

As $x^0$ is a local solution of $(P)$, the same point is also a local solution of the problem

$$\min_{x \in K_1} \mathcal{L}(x, u, w) = f(x).$$

But, thanks to the Karush-Kuhn-Tucker conditions, it holds $\nabla_x \mathcal{L}(x^0, u, w) = 0$. Applying Theorem 9, we have the following result.

**Theorem 10.** Let $x^0 \in K$ be a local solution of $(P)$ and let $(x^0, u, w)$ be a triplet satisfying the related Karush-Kuhn-Tucker conditions. Then we have

$$y^\top \nabla_x^2 \mathcal{L}(x^0, u, w) y \geqq 0, \quad \forall y \in T(K_1; x^0).$$

Theorem 10 may be viewed as a relaxed version of Theorem 6, as it holds under *any* first-order constraint qualification. On the other hand, it is not difficult to prove that it holds

$$T(K_1; x^0) \subset Z(x^0).$$

Obviously, if it holds

$$T(K_1; x^0) = Z(x^0) \tag{1}$$

we get just the thesis of Theorem 6. A sufficient condition to have (1) is (LICQ), as shown by Theorem 6, but a more general sufficient condition is (SMFCQ). We recall that (SMFCQ) is also a first-order constraint qualification and that it holds

$$(LICQ) \Longrightarrow (SMFCQ) \Longrightarrow (MFCQ).$$

Hence, similarly to the (LICQ) condition, the (SMFCQ) condition is both a first-order and a second-order constraint qualification for the strong second-order necessary optimality conditions. This was observed also by Kyparisis (1985). We have already recalled that (MFCQ) has not the said properties. Theorem 10 appears also in the book of Forst and Hoffmann (2010) and in the book of Güler (2010).

**Theorem 11.** Let $x^0$ be a local minimum for $(P)$ and let the (SMFCQ) condition be satisfied at $x^0$. Then, in addition to the (KKT) conditions, it holds

$$T(K_1; x^0) = Z(x^0).$$

**Proof.** We give only a suggestion of the proof and leave the details to the willing reader. First we note that the "linearizing cone" of $K_1$ is just the cone $Z(x^0)$. Then there is only to fit to the present case the proof of the implication

$$(MFCQ) \Longrightarrow (\text{Abadie CQ}).$$

See, e. g., Bazaraa and Shetty (1976), Bomze (2016), Giorgi (2018). $\square$

Other conditions assuring that the strong second-order necessary optimality conditions hold for $(P)$ are:

- All constraints functions $g$ and $h$ are affine.
- The objective function $f$ and the constraints $g_i$, $i \in M$, are convex, the constraints $h_j$, $j \in P$, are affine and $\bar{x} \in K$ satisfies the *Slater constraint qualification.* See, e. g., Bazaraa, Sherali and Shetty (2006), Bonnans and Shapiro (2000).

Another way to obtain strong second-order necessary optimality conditions for $(P)$, with the use of any first-order constraint qualification (and hence also of (MFCQ)) is presented by McCormick (1967) in his pioneering paper on second-order optimality conditions for a mathematical programming problem. This author makes use of an appropriate "second-order constraint qualification". This approach is presented also by Fiacco and McCormick (1968a) and by Luenberger and Ye (2008). All these authors make reference to the cone $Z_1(x^0)$, whereas McCormick (1976) makes reference to the larger cone $Z(x^0)$. We recall that the two cones coincide when the *Strict Complementarity Slackness Condition* holds at $x^0$, i. e. it holds $u_i > 0$ for every $i \in I(x^0)$, in the (KKT) conditions.

- The *McCormick Second-Order Constraint Qualification* holds at $x^0 \in K$ if for all $z \in Z(x^0)$, $z$ is the tangent of a twice differentiable arc $\alpha(\theta)$ where for some $\theta_1 > 0$,

$$\begin{aligned} g_i(\alpha(\theta)) &= 0, \text{ for } \theta \in [0, \theta_1], \ \forall i \in I^+(x^0, u); \\ g_i(\alpha(\theta)) &\leqq 0, \text{ for } \theta \in [0, \theta_1], \ \forall i \in I(x^0) \diagdown I^+(x^0, u); \\ h_j(\alpha(\theta)) &= 0, \text{ for } \theta \in [0, \theta_1], \ \forall j = 1, ..., p. \end{aligned}$$

We have the following result.

**Theorem 12 (McCormick).** Let $x^0 \in K$ be a local solution of $(P)$ and suppose that there exist vectors $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ such that the Karush-Kuhn-Tucker conditions are satisfied at $x^0$. Further, suppose that the McCormick Second-Order CQ holds at $x^0$. Then we have

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \ \forall z \in Z(x^0).$$

It can be proved that the (LICQ) condition implies the McCormick CQ (the same is true for (SMFCQ)). So, also for this approach, the (LICQ) and the (SMFCQ) conditions are both a first-order CQ and a second-order CQ. However, the (MFCQ) condition does not imply the McCormick Second-Order CQ. Fiacco and McCormick (1968a) remark that the Kuhn-Tucker (first-order) constraint qualification does not imply the McCormick Second-Order CQ.. Also the viceversa does not hold: see Giorgi (2018). Finally, we remark that if $g$ and $h$ are linear affine functions, the McCormick Second-Order CQ is automatically satisfied.

Another approach to obtain second-order necessary conditions in terms of Fritz John conditions (therefore without any constraint qualification) and in terms of the contingent cone is presented (as a generalization of a result of Hestenes (1966)) in an unpublished paper by Fiacco and McCormick (1968b). This result has been subsequently considered by McCormick

(1976, 1983) and by Bector, Chandra and Dutta (2005), who, however, make some unnecessary assumptions. See also Giorgi and Zuccotti (2008). We recall that the cone $K_1$ has been defined as:

$$K_1 = \left\{ x \in K : g_i(x) = 0, \ \forall i \in I^+(x^0, u) \right\}$$

where $I^+(x^0, u) = \{ i \in I(x^0) : u_i > 0 \}$.

**Theorem 13.** Let $x^0 \in K$ be a local minimum point for $(P)$. Then, in addition to the Fritz John conditions for $(P)$, i. e. there exist multipliers $u_0, u_i$, $i = 1, ..., m$; $w_j$, $j = 1, ..., p$, *not all zero,* such that

$$\nabla_x \mathcal{L}_1(x^0, u_0, u, w) = 0;$$

$$u_i g_i(x^0) = 0, \ i = 1, ..., m;$$

$$u_0 \geqq 0, \ u_i \geqq 0, \ i = 1, ..., m,$$

where

$$\mathcal{L}_1(x, u_0, u, w) = u_0 f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x),$$

one has

$$z^\top \nabla_x^2 \mathcal{L}_1(x^0, u_0, u, w) z \geqq 0, \quad \forall z \in T(K_1; x^0).$$

The reader will note the differences and the similarities between the previous result and the one of Theorem 10. As McCormick himself (McCormick (1976, 1983)) remarks, Theorem 13 is limited in applications, since the multiplier $u_0$ might be equal to zero (the usual handicap of the Fritz John conditions!) and, above all, since it would be extremely difficult to valuate the contingent cone $T(K_1; x^0)$. This is true obviously also for Theorem 10 and for all the other results where the contingent cone appears.

When the *McCormick Second-Order Constraint Qualification* holds, it can be shown that the cone $T(K_1; x^0)$ and the critical cone $Z(x^0)$ are the same. If (LICQ) or also (SMFCQ) holds at $x^0$, then not only

$$T(K_1; x^0) = Z(x^0) = C(x^0),$$

but we can also choose $u_0 = 1$ in the (FJ) conditions.

Another approach to get second-order necessary optimality conditions for $(P)$ has been extensively considered by Ben-Tal (1980) but, prior to Ben-Tal, also by Hettich and Jongen (1977). This approach leads to what we may call *weak second-order necessary optimality conditions for* $(P)$. In all previous conditions the multipliers $(u, w)$ of the (KKT) conditions or the multipliers $(u_0, u, w)$ of the Fritz John conditions were considered *fixed,* for every vector of critical directions or belonging to the contingent cone $T(K_1; x^0)$. The Ben-Tal approach *does not* guarantee that the *same* multiplier vectors can be taken such that the corresponding second-order necessary optimality conditions are satisfied. Usually, under the said approach, the following cone is considered:

$$C_1(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla f(x^0)z \leqq 0; \\ \nabla g_i(x^0)z \leqq 0, \ i \in I(x^0); \\ \nabla h_j(x^0)z = 0, \ j = 1, ..., p. \end{array} \right\}$$

We may call $C_1(x^0)$ the *extended critical cone.* Again we point out that there are not standard denominations in the literature: for example, Andreani, Behling, Haeser and Silva (2017) call $C_1(x^0)$ *strong critical cone at* $x^0$ and call $C(x^0)$ *weak critical cone at* $x^0$.

Between $C(x^0)$ and $C_1(x^0)$, besides the obvious fact that it holds $C(x^0) \subset C_1(x^0)$, there are the relationships specified in the following result (see Giorgi and Zuccotti (2008), Hettich and Jongen (1977)).

**Theorem 14.** Let $x^0$ be a local minimum point for $(P)$. Then there exist numbers $u_0$, $u_i$ $(i = 1, ..., m)$, and $w_j$ $(j = 1, ..., p)$, *not all zero,* such that the (FJ) conditions are satisfied at $x^0$. Moreover, $u_0 > 0$ if and only if $C(x^0) = C_1(x^0)$.

We give first the weak second-order necessary optimality conditions in terms of the (FJ) conditions, therefore without any constraint qualification.

**Theorem 15.** Let $x^0 \in K$ be a local minimum point for $(P)$; then for every $z \in C_1(x^0)$, there exist multipliers $(u_0, u, w)$ with $(u_0, u) \geqq 0$ and $u_0, u, w$ not all zero, such that the (FJ) conditions hold and

$$z^\top \nabla_x^2 \mathcal{L}_1(x^0, u_0, u, w)z \geqq 0.$$

Moreover, if $C_1(x^0) \neq C(x^0)$, $u_0$ is always equal to zero.

See, e. g., Hettich and Jongen (1977), Ben-Tal (1980), Still and Streng (1996), Bonnans and Shapiro (2000). We draw the reader's attention to the different structure of Theorem 15 with respect to the strong second-order necessary optimality conditions. In Theorem 15 the multipliers $(u_0, u, w)$ appearing in the Lagrangian function of the quadratic form are not required to be fixed. Hettich and Jongen (1977) and Ben-Tal (1980) provide examples where there exist no fixed multipliers for any critical direction. The conditions of Theorem 15 can be written also as

$$\sup_{(u_0,u,w)\in\Lambda_0(x^0)} z^\top \nabla_x^2 \mathcal{L}_1(x^0, u_0, u, w)z \geqq 0, \quad \forall z \in C_1(x^0).$$

**Remark 5.** A consequence of Theorem 15 is the validity of the following relation:

$$u_i \nabla g_i(x^0)z = 0, \ i = 1, ..., m, \ z \in C_1(x^0).$$

Indeed, if we multiply the two members of the first Fritz John relation by $z \in C_1(x^0)$ we obtain

$$u_0 \nabla f(x^0)z + \sum_{i=1}^{m} u_i \nabla g_i(x^0)z + \sum_{j=1}^{p} w_j \nabla h_j(x^0)z = 0.$$

If $u_0 = 0$ obviously we have $u_0 \nabla f(x^0)z = 0$ and the same is true if $u_0 > 0$, as in this case $C_1(x^0) = C(x^0)$. On the other hand, we have always

$$\sum_{j=1}^{p} w_j \nabla h_j(x^0)z = 0,$$

being $z \in C_1(x^0)$. Hence it will hold

$$\sum_{i=1}^{m} u_i \nabla g_i(x^0)z = 0, \;\; z \in C_1(x^0),$$

but, being $u_i \geqq 0$, $\forall i = 1, ..., m$, $u_i = 0$ for $i \notin I(x^0)$, and $\nabla g_i(x^0)z \leqq 0$, $\forall i \in I(x^0)$, we get

$$u_i \nabla g_i(x^0)z = 0, \;\; i = 1, ..., m.$$

The usual drawback of Theorem 15 is that the first multiplier $u_0$ can be zero. As already remarked, the Mangasarian-Fromovitz constraint qualification *does not assure* strong second-order necessary optimality conditions, however, (MFCQ) assures the validity of Theorem 15, with $u_0 > 0$.

**Theorem 16.** Assume that $x^0 \in K$ is a local minimum point for $(P)$ that satisfies the (MFCQ) condition. Then $\Lambda(x^0)$ is nonempty, convex and compact. We have that for every $z \in C(x^0)$ there exist multipliers $(u, w) \in \Lambda(x^0)$ such that

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w)z \geqq 0,$$

i. e. (being $\Lambda(x^0)$ compact)

$$\max_{(u,w)\in\Lambda(x^0)} z^\top \nabla_x^2 \mathcal{L}(x^0, u, w)z \geqq 0, \quad \forall z \in C(x^0).$$

See, e. g., Ben-Tal (1980), Still and Streng (1996).

Baccari (2004) and Baccari and Trad (2004) give some sufficient conditions for the (MFCQ) to assure strong second-order necessary optimality conditions. A constraint qualification implied by (MFCQ) and therefore weaker than (MFCQ) but which assures the thesis of Theorem 16, is provided by Ben-Tal (1980). See also Still and Streng (1996).

**Definition 4. (Ben-Tal Second-Order Constraint Qualification).** Let $x^0 \in K$. We say that at $x^0$ the Ben-Tal SOCQ is fulfilled if:

$(i)$ $\nabla h_j(x^0)$, $\forall j \in P$, are linearly independent.

$(ii)$ There exists a vector $z \in \mathbb{R}^n$, $z \neq 0$, such that for any nonzero vector $y \in L(x^0)$, where

$$L(x^0) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0)y \leqq 0, \;\; i \in I(x^0); \;\; \nabla h_j(x^0)y = 0, \;\; j = 1, ..., p \right\}$$

is the *linearizing cone at* $x^0$, it holds

$$\begin{aligned} y^\top \nabla_x^2 g_i(x^0)y + \nabla g_i(x^0)z &< 0, \quad \forall i \in I^*(x^0, y); \\ y^\top \nabla_x^2 h_j(x^0)y + \nabla h_j(x^0)z &= 0, \quad \forall j \in P, \end{aligned}$$

where

$$I^*(x^0, y) = \left\{ i \in I(x^0) : \nabla g_i(x^0)y = 0, \ y \in L(x^0) \right\}.$$

**Remark 6.** It holds, as already observed,

$$(MFCQ) \Longrightarrow (\text{Ben-Tal } SOCQ).$$

The converse does not hold. Moreover, the Ben-Tal SOCQ is also a first-order constraint qualification. In other words, under this constraint qualification the Fritz John conditions hold at a local optimal point $x^0$ for $(P)$ with $u_0 > 0$, i. e. $u_0 = 1$.

**Theorem 17.** Suppose that at $x^0 \in K$, local minimum point for $(P)$, with $L(x^0) \neq \{0\}$, the Ben-Tal SOCQ is satisfied. Then at $x^0$ the (KKT) conditions hold and in addition the weak second-order necessary optimality conditions of Theorem 16 hold.

Another constraint qualification, introduced by Janin (1984) as a first-order constraint qualification useful in the study of sensitivity and stability properties of a nonlinear programming problem, is the *Constant Rank Condition* (CRC). This constraint qualification was recognized by Andreani, Echagüe and Schuverdt (2010) to be also a second-order constraint qualification. Indeed, these authors obtain, under (CRC), a strong second-order necessary optimality condition for $(P)$.

**Theorem 18.** Let $x^0 \in K$ be a local solution for $(P)$ and let the (CRC) constraint qualification be satisfied at $x^0$. Then $\Lambda(x^0) \neq \varnothing$ and for *every* pair $(u, w) \in \Lambda(x^0)$ it holds

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w)z \geqq 0, \quad \forall z \in C(x^0).$$

We recall that in the previous result it holds $C(x^0) = Z(x^0)$. In the same paper the above authors establish another strong second-order optimality condition under the *Weak Constant Rank Condition.*

**Definition 5.** Let $x^0 \in K$; we say that at $x^0$ the *Weak Constant Rank Condition* (WCRC) holds if there is a neighborhood $U$ of $x^0$ such that the set

$$\left\{ \nabla g_i(x), \ i \in I(x^0) \right\} \cup \left\{ \nabla h_j(x), \ j = 1, ..., p \right\}$$

has the same rank for all $x \in U$.

Andreani, Martinez and Schuverdt (2007) prove that (WCRC) is *not* a first-order constraint qualification. The same authors prove the following result.

**Theorem 19.** Let $x^0 \in K$ be a local solution for $(P)$ and let $\Lambda(x^0) \neq \varnothing$ (i. e. at $x^0$ the (KKT) conditions are satisfied). Let (WCRC) be satisfied at $x^0$. Then, for *every* pair $(u, w) \in \Lambda(x^0)$ it holds

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z_1(x^0). \tag{2}$$

Hence, any constraint qualification that ensures $\Lambda(x^0) \neq \varnothing$ in combination with (WCRC), guarantees that (2) holds at a local minimizer $x^0$ of $(P)$. Minchenko and Stakhovski (2011) proved that under the *Relaxed Constant Rank Constraint Qualification* (RCRCQ) we have both $\Lambda(x^0) \neq \varnothing$ and the validity of (WCRC):

- The (RCRCQ) condition holds at $x^0 \in K$ if there exists a neighborhood $U$ of $x^0$ such that for every index set $I \subset I(x^0)$, the set

$$\{\nabla g_i(x),\ i \in I\} \cup \{\nabla h_j(x),\ j = 1, ..., p\}$$

has the same rank for all $x \in U$.

# 3. Other Approaches to Second-Order Necessary Conditions Under Various Constraint Qualifications and Assumptions

In the present section we consider some other approaches to second-order necessary optimality conditions, under various constraint qualifications and assumptions different from the ones previously considered.

**A)** Except for the McCormick second-order constraint qualification and the Ben-Tal second-order constraint qualification, all constraint qualifications and regularity conditions seen previously involve only first-order derivatives. Several authors (e. g. Aghezzaf and Hachimi (1999), Bigi and Castellani (2000), Cambini, Martein and Vlach (1999), Bonnans and Shapiro (2000), Bonnans, Cominetti and Shapiro (1999), Cominetti (1990), Constantin (2006, 2016), Di (1996), Castellani and Pappalardo (1996), He and Sun (2011), Jiménez and Novo (2003, 2004), Giorgi, Jiménez and Novo (2010), Penot (1994, 1998, 2000), Ward (1993)) introduce second-order analogues of the said definitions and concepts, in order to obtain second-order optimality conditions under general assumptions. We have seen the definition of *Bouligand tangent cone or contingent cone* $T(S; x^0)$ to a set $S \subset \mathbb{R}^n$ at $x^0 \in S$ (or also at $x^0 \in cl(S)$), as a first-order local cone approximation of $S$ at $x^0$. The following well-known second-order approximation of a set is a true second-order extension of the contingent cone.

**Definition 6.** Let $S \subset \mathbb{R}^n$. The *second-order contingent set* of $S$ at $x^0 \in cl(S)$ in the direction $v \in \mathbb{R}^n$ is the set

$$T^2(S; x^0; v) = \left\{ w \in \mathbb{R}^n : \exists t_n \longrightarrow 0^+,\ \exists w^n \longrightarrow w \text{ such that } x^0 + t_n v + \frac{1}{2} t_n^2 w^n \in S \right\}.$$

This set can be equivalently described as follows:

$$T^2(S; x^0; v) = \left\{ w \in \mathbb{R}^n : \forall \delta > 0,\ \exists t \in (0, \delta),\ \exists w' \in N(w, \delta) \text{ such that } x^0 + tv + \frac{1}{2} t^2 w' \in S \right\};$$

$$T^2(S;x^0;v)=\left\{w\in\mathbb{R}^n:\exists x^n\in S,\ \exists t_n\longrightarrow 0^+ \text{ such that } x^n=x^0+t_nv+\frac{1}{2}t_n^2w+o(t_n^2)\right\},$$

where $o(t_n^2)$ is a vector satisfying $(\|o(t_n^2)\|\diagup t_n^2)\longrightarrow 0$;

$$T^2(S;x^0;v)=\limsup_{t\longrightarrow 0^+}\frac{S-x^0-tv}{\frac{1}{2}t^2}.$$

In the last expression the limit is intended in the Kuratowski sense (see Aubin and Frankowska (1990)).

We note that when $v=0$ the second-order contingent set collapses into the contingent cone $T(S;x^0)$. It is worth stressing that in general the second-order contingent set is not a cone and that it does not preserve convexity. Furthermore, it is easy to show that $T^2(S;x^0;v)=\varnothing$ when $v\notin T(S;x^0)$, but the viceversa does not hold. Therefore, the second-order contingent set may be empty even if the direction $v$ is chosen in the contingent cone. Therefore we assume $T^2(S;x^0;v)\neq\varnothing$.

The set $T^2(S;x^0;v)$ however preserves some properties of the contingent cone: for instance, it is closed and is isotone, i. e. if $S_1\subset S_2$ and $x^0\in cl(S_1)$, then $T^2(S_1;x^0;v)\subset T^2(S_2;x^0;v)$. If $S$ is a *polyhedral set,* then we have

$$T^2(S;x^0;v)=T(T(S;x^0);v),$$

therefore in this case $T^2(S;x^0;v)$ is a convex cone (see Ruszczynski (2006)).

A first result concerning second-order necessary conditions in terms of $T^2(S;x^0;v)$ takes into consideration problem $(P_1)$ :

$$(P_1):\qquad \min_{x\in S}f(x).$$

This result is due to Penot (1994); for the reader's convenience we give its proof, as it is quite simple.

**Theorem 20.** Assume that $x^0$ is a local solution of $(P_1)$. Then, for every $v\in T(S;x^0)$ such that $\nabla f(x^0)v=0$, we have

$$\nabla f(x^0)w+v^\top\nabla^2f(x^0)v\geqq 0,\quad \forall w\in T^2(S;x^0;v).$$

**Proof.** Let $v\in T(S;x^0)$ with $\nabla f(x^0)v=0$ and $w\in T^2(S;x^0;v)$. By the definition of $T^2(S;x^0;v)$, there are sequences $\{x^n\}\subset S$ and $t_n\longrightarrow 0^+$ such that $x^n=x^0+t_nv+\frac{1}{2}t_n^2w+o(t_n^2)$, hence the Taylor expansion yields

$$f(x^n)=f(x^0)+t_n\nabla f(x^0)v+\frac{1}{2}t_n^2(\nabla f(x^0)w+v^\top\nabla^2f(x^0)v)+o(t_n^2).$$

The result follows dividing by $t_n^2$ and taking the limit for $n\longrightarrow+\infty$, as $x^n\longrightarrow x^0$, $o(t_n^2)\diagup t_n^2\longrightarrow 0$, $\nabla f(x^0)v=0$ and $f(x^0)\leqq f(x^n)$ for sufficiently large $n$. □

In order to obtain a similar statement for problem $(P)$, we define the *second-order linearizing set of* $K$ *at* $x^0 \in K$ *in the direction* $v$ :

$$L^2(x^0; v) = \left\{ \begin{array}{l} w \in \mathbb{R}^n : \nabla g_i(x^0)w + v^\top \nabla^2 g_i(x^0)v \leqq 0, \quad \forall i \in I^*(x^0; v); \\ \nabla h_j(x^0)w + v^\top \nabla^2 h_j(x^0)v = 0, \quad j = 1, ..., p \end{array} \right\},$$

where, if $I(x^0) \neq \varnothing$, $I^*(x^0; v) = \{i \in I(x^0) : \nabla g_i(x^0)v = 0\}$.

Following some authors (e. g. Jiménez and Novo (2003, 2004), Bigi and Castellani (2000), Kawasaki (1988), Aghezzaf and Hachimi (1999)), we introduce for $(P)$ some constraint qualifications in terms of the definitions previously given. Let $K$ be nonempty, $x^0 \in K$, $v \in L(x^0)$. We recall that $L(x^0)$ is the *linearizing cone* of $K$ at $x^0$ :

$$L(x^0) = \left\{ v \in \mathbb{R}^n : \nabla g_i(x^0)v \leqq 0, \ i \in I(x^0); \quad \nabla h_j(x^0)v = 0, \ j = 1, ..., p \right\}.$$

Then:

(1) The pair $(x^0; v)$ satisfies the *second-order linear independence constraint qualification* (SOLICQ) if the vectors

$$\left\{ \nabla g_i(x^0), \ i \in I^*(x^0; v), \ \nabla h_j(x^0), \ j = 1, ..., p \right\}$$

are linearly independent.

(2) The *second-order Mangasarian-Fromovitz constraint qualification or second-order Ben-Tal constraint qualification* is satisfied at $x^0$ if

$(i)$ The vectors $\{\nabla h_j(x^0), \ j = 1, ..., p\}$ are linearly independent;

$(ii)$ It holds $L^2_<(x^0; v) \neq \varnothing$, where

$$L^2_<(x^0; v) = \left\{ \begin{array}{c} w \in \mathbb{R}^n : \nabla g_i(x^0)w + v^\top \nabla^2 g_i(x^0)v < 0, \quad \forall i \in I^*(x^0; v); \\ \nabla h_j(x^0)w + v^\top \nabla^2 h_j(x^0)v = 0, \quad j = 1, ..., p \end{array} \right\}.$$

(We have previously already considered the definition of second-order Ben-Tal CQ).

(3) The *second-order Abadie constraint qualification* holds at $x^0$ in the direction $v \in L(x^0)$ if

$$T^2(K; x^0; v) = L^2(x^0; v) \neq \varnothing.$$

**Theorem 21.**

$(i)$ Let $x^0 \in K$ and $v \in L(x^0)$. It holds at $(x^0, v)$, $\forall v \in L(x^0)$ :

a) (First-order LICQ) $\Longrightarrow$ (SOLICQ).

b) (First-order MFCQ) $\Longrightarrow$ (Second-order MFCQ).

c) (SOLICQ) $\Longrightarrow$ (Second-order MFCQ) $\Longrightarrow$ (Second-order Abadie CQ).

$(ii)$ Let $x^0 \in K$. If $T^2(K; x^0; v) \neq \varnothing$, $\forall v \in L(x^0)$, then

$$L(x^0) = T(K; x^0),$$

i. e. the first-order Abadie constraint qualification holds. (Recall that if $T^2(K; x^0; v) \neq \varnothing$, then $v \in T(K; x^0)$).

**Proof.** See Giorgi (2019).

The following result can be found, e. g., in Bigi and Castellani (2000), referred to a vector optimization problem.

**Theorem 22.** Let us suppose that the second-order Abadie CQ holds for every vector $v \in C(x^0) = Z(x^0)$. If $x^0 \in K$ is a local solution of $(P)$, then for each vector $v \in C(x^0)$ there exist multiplier vectors $u \in \mathbb{R}^m_+$ and $w \in \mathbb{R}^p$ such that the Karush-Kuhn-Tucker conditions are satisfied at $x^0$ and moreover the weak second-order necessary conditions hold at $x^0$ :

$$v^\top \nabla^2_x \mathcal{L}(x^0, u, v) v \geqq 0.$$

Bigi and Castellani (2000) remark that the assumptions of Theorem 22 can be weakened by requiring that the second-order Abadie CQ holds only for some directions $v \in C(x^0)$. Obviously, in this case the second-order optimality conditions will hold only for those directions.

We note moreover, that in fact the assumptions of Theorem 22 can be further weakened by assuming the *second-order Gould-Tolle-Guignard constraint qualification,* i. e. for each $v \in L(x^0)$ it holds

$$(T^2(K; x^0; v))^* = (L^2(x^0, v))^*.$$

We note finally that the second-order versions of the Abadie CQ and of the Gould-Tolle-Guignard CQ are in fact stronger than their first-order counterparts; this is not true for the second-order Mangasarian-Fromovitz CQ with respect to the first-order Mangasarian-Fromovitz CQ. For other considerations on the results of Theorem 22 (referred to a vector optimization problem) see the paper of Kim and Tuyen (2018) where an error of Rizvi and Nasser (2006) is corrected.

**Remark 7.** A similar approach has been adopted in a pioneering paper of Kawasaki (1988a), where the *lexicographic order* is used:

- for any two-dimensional vectors $p = (p, p_2)^\top$ and $q = (q_1, q_2)^\top$ the notation $p \leqq_{lex} q$ means that $p_1 < q_1$ holds or $p_1 = q_1$ and $p_2 \leqq q_2$ hold. Similarly, $p <_{lex} q$ means that $p_1 < q_1$ holds or $p_1 = q_1$ and $p_2 < q_2$ hold.

Some authors (Penot (1998, 2000), Cambini, Martein and Vlach (1999)) have introduced another local second-order approximation of a set $S \subset \mathbb{R}^n$, the so-called *asymptotic second-order tangent cone to* $S$ at $(x^0, v)$ :

$$T''(S; x^0; v) = \left\{ \begin{array}{l} w \in \mathbb{R}^n : \exists (t_n, r_n) \longrightarrow (0^+, 0^+),\ \exists w^n \longrightarrow w \text{ such that } \frac{t_n}{r_n} \longrightarrow 0 \\ \text{and } x^0 + t_n v + \frac{1}{2} t_n r_n w_n \in S, \quad \forall n \in \mathbb{N} \end{array} \right\}.$$

See also Jiménez and Novo (2004) and Giorgi, Jiménez and Novo (2010). This cone can be equivalently defined as:

$$T''(S; x^0; v) = \left\{ \begin{array}{l} w \in \mathbb{R}^n : \exists (t_n, r_n) \longrightarrow (0^+, 0^+),\ \exists w^n \longrightarrow w \text{ such that } \frac{t_n}{r_n} \longrightarrow 0 \\ \text{and } x^0 + t_n v + \frac{1}{2} t_n^2 r_n^{-1} w_n \in S, \quad \forall n \in \mathbb{N} \end{array} \right\}.$$

$$T''(S;x^0;v) = \left\{ \begin{array}{l} w \in \mathbb{R}^n : \exists(\alpha_n, \beta_n) \longrightarrow (+\infty, +\infty),\ \exists x^n \in S \text{ such that } \frac{\beta_n}{\alpha_n} \longrightarrow 0 \\ \text{and } \beta_n(\alpha_n(x^n - x^0) - v) \longrightarrow w \end{array} \right\}.$$

It can be proved that $T''(S;x^0;v)$, is contained in $cl\,\{cone\,[cone(S-x^0)-v]\}$. In particular, if $S$ is a convex set, then

$$T^2(S;x^0;v) \subset T''(S;x^0;v) = cl\,\left\{cone\,\left[cone(S-x^0)-v\right]\right\}.$$

In general, there is no inclusion relation between $T^2(S;x^0;v)$ and $T''(S;x^0;v)$. It can be remarked that $T''(S;x^0;v)$ is a closed cone and that if $v \notin T(S;x^0)$, then $T''(S;x^0;v) = \varnothing$. If $v = 0$, then $T^2(S;x^0;v) = T''(S;x^0;v) = T(S;x^0)$.

Another second-order approximation of a set $S \subset \mathbb{R}^n$ is the *second-order projective tangent cone* of $S$ at $x^0 \in S$, introduced by Penot (1998):

$$\hat{T}^2(S;x^0;v) = \left\{ \begin{array}{l} (w,r) \in \mathbb{R}^{n+1} :\ \exists(t_n, r_n) \longrightarrow (0^+, 0^+),\ \exists w^n \longrightarrow w \\ \text{such that } (t_n/r_n) \longrightarrow r,\ x^0 + t_n v + \frac{1}{2} t_n r_n w_n \in S,\ \forall n \in \mathbb{N} \end{array} \right\}.$$

Some authors (Penot (1998, 2000), Cambini, Martein and Vlach (1999), Jiménez and Novo (2003, 2004), Constantin (2006, 2016)) use both the cones $T''(S;x^0;v)$ and $\hat{T}^2(S;x^0;v)$ to obtain second-order optimality conditions. For example, we have the following result for a nonlinear programming problem with a set constraint, i. e.

$$(P_1): \qquad \min f(x), \quad x \in S,$$

where $S \subset \mathbb{R}^n$ is any nonempty subset of $\mathbb{R}^n$.

**Theorem 23.** Let $x^0$ be a local minimum of $f$ on $S$. Let $v \in T(S;x^0)$ with $\nabla f(x^0)v = 0$. Then we have

(*i*) $\nabla f(x^0)w + rv^\top \nabla^2 f(x^0)v \geqq 0$, $\forall (w,r) \in \hat{T}^2(S;x^0;v)$.

(*ii*) $\nabla f(x^0)w \geqq 0$, $\forall w \in T''(S;x^0;v)$.

**B)** In recent papers on second-order optimality conditions some authors obtain strong second-order necessary conditions by imposing some assumptions besides the usual Mangasarian-Fromovitz constraint qualification. For example, Baccari (2004) and Baccari and Trad (2004) have proved the following results.

**Theorem 24.** Let $x^0$ be a local optimal solution of problem $(P)$ and let (MFCQ) be satisfied. Assume further that one of the following conditions hold:

1) $n \leqq 2$;

2) $card(I(x^0)) \leqq 2$.

Then, there exists a pair $(u,w) \in \Lambda(x^0)$ such that

$$z^\top \nabla_x^2 \mathcal{L}(x^0,u,w)z \geqq 0, \quad \forall z \in C(x^0).$$

**Theorem 25.** Let $x^0$ be a local optimal solution of problem $(P)$ and assume that the following conditions hold:

$(i)$ The set of Lagrange multipliers $\Lambda(x^0)$ is a closed and bounded line segment.

$(ii)$ There exists at most only one index $i_0 \in I(x^0)$ such that for the related Karush-Kuhn-Tucker multipliers it holds $u_{i_0} = 0$.

Then, the thesis of Theorem 24 holds.

Obviously, if the *Strict Complementarity Slackness Condition holds,* then $(ii)$ of Theorem 25 is satisfied.

Another result of Baccari and Trad (2004) on these types of questions has been generalized by Behling, Haeser, Ramos and Viana (2018). Let $J(x^0)$ be the $(q+p) \times n$ matrix formed by the gradients $\nabla g_i(x^0)$, $i \in I(x^0)$, $\nabla h_j(x^0)$, $j = 1, ..., p$. Therefore $q$ is the cardinality of $I(x^0)$.

**Theorem 26.** Let $x^0$ be a local optimal solution of problem $(P)$, such that the (MFCQ) holds and the rank of $J(x^0) \in \mathbb{R}^{(q+p)\times n}$ is $(n-1)$. Then, there exists a pair $(u, w) \in \Lambda(x^0)$ such that

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z_1(x^0),$$

where $Z_1(x^0)$ is the *critical subspace* at $x^0 \in K$ :

$$Z_1(x^0) = \left\{ z \in \mathbb{R}^n : \nabla g_i(x^0) z = 0, \ i \in I(x^0); \ \nabla h_j(x^0) z = 0, \ j = 1, ..., p \right\}.$$

Behling, Haeser, Ramos and Viana (2018), under a technical assumption (Assumption 3.1 in Behling and others (2018)), improve the previous results and prove a conjecture previously made by Andreani, Martinez and Schuverdt (2007).

In an extended technical paper Behling and others (2017) show that some related results of Shen, Xue and An (2015) and Minchenko and Leschov (2016) are not correct.

Another result concerning second-order strong optimality conditions is due to Daldoul and Baccari (2009). These authors consider a quadratic approximation of problem $(P)$, i. e.

$$(QP): \begin{cases} \min \left\{ f(x^0) + \nabla f(x^0)(x-x^0) + \frac{1}{2}(x-x^0)^\top \nabla^2 f(x^0)(x-x^0) \right\} \\ \text{subject to :} \\ \qquad g_i(x^0) + \nabla g_i(x^0)(x-x^0) + \frac{1}{2}(x-x^0)^\top \nabla^2 g_i(x^0)(x-x^0) \leqq 0, \ i = 1, ..., m, \\ \qquad h_j(x^0) + \nabla h_j(x^0)(x-x^0) + \frac{1}{2}(x-x^0)^\top \nabla^2 h_j(x^0)(x-x^0) = 0, \ j = 1, ..., p, \end{cases}$$

where $x^0$ is a local minimizer for problem $(P)$. Let us denote by (WCRQP) the *weak constant rank condition,* referred to $(QP)$. See Definition 5 for (WCR), referred to $(P)$. It can be seen that (WCRQP) holds if the Jacobian matrix of active constraints has full rank.

**Theorem 27 (Daldoul and Baccari).** Let $x^0$ be a local solution of $(P)$, let the (MFCQ) condition be verified, together with the (WCRQP) condition. Then the thesis of Theorem 26 holds.

**C)** Andreani, Behling, Haeser and Silva (2017) present various second-order necessary optimality conditions in terms of Abadie-type (first-order) constraint qualifications. In particular, we report here their Theorem 3.3, previously asserted (without proof) by Bazaraa, Sherali and Shetty (2006).

**Theorem 28.** Let $x^0$ be a local minimum of $(P)$ and $(u, w) \in \mathbb{R}^m_+ \times \mathbb{R}^p$ an associated Karush-Kuhn-Tucker multipliers vector. Let $I^+(x^0, u)$ be the set

$$I^+(x^0, u) = \left\{i \in I(x^0) : u_i > 0\right\}.$$

Let us consider the set

$$K_1 = \left\{x \in K : g_i(x) = 0, \ \forall i \in I^+(x^0, u\right\}.$$

If The Abadie constraint qualification, referred to $K_1$, holds at $x^0$, then the strong second-order optimality conditions hold, with multipliers $(u, w)$ and with respect to the cone $Z(x^0) = C(x^0)$.

This is in fact an immediate consequence of Theorem 10; it is sufficient to note that the *linearizing cone* of $K_1$ is nothing but the critical cone $Z(x^0)$, coinciding with $C(x^0)$, as the Karush-Kuhn-Tucker conditions hold at $x^0$.

Similar questions have been treated also by Bomze (2016) who, however, uses a constraint qualification more general than the Abadie one. Bomze introduces an apparently new constraint qualification, he calls *reflected Abadie CQ.* Given $(P)$ and a point $x^0 \in K$ we say that $g$ and $h$ satisfy the reflected Abadie CQ at $x^0$ if

$$L(x^0) \subset T(K; x^0) \cup \left[-T(K; x^0)\right].$$

In the same paper Bomze (2016) remarks that the usual Abadie CQ implies the reflected Abadie CQ (and the Gould-Tolle-Guignard CQ), but in general there are no inclusion relations between the reflected Abadie CQ and the Gould-Tolle-Guignard CQ. Moreover, it must be stressed that the constraint qualification introduced by Bomze really *does not assure* the existence of Karush-Kyhn-Tucker multipliers at an optimal point of $(P)$, i. e. it is *not* a first-order constraint qualification.

Bomze then considers the set

$$K_1 = \left\{x \in K : g_i(x) = 0, \ \forall i \in I^+(x^0, u\right\}$$

and proves the following result.

**Theorem 29.** Let $x^0$ be a local solution of $(P)$ and let the *reflected Abadie CQ* be satisfied with respect to $K_1$. Let the Karush-Kuhn-Tucker conditions for $(P)$ hold at $x^0$, with multipliers vectors $(u, w)$. Then we have

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z(x^0) = C(x^0).$$

**D)** A recent and interesting approach to second-order necessary optimality conditions has been proved by Andreani, Haeser, Ramos and Silva (2017), on the trail of what previously done with reference to first-order optimality conditions, i. e. the *approximate (or sequential or asymptotic) Karush-Kuhn-Tucker conditions,* introduced by Andreani, Haeser and Martinez (2011) and in the unpublished paper of Fiacco and McCormick (1968b). See Giorgi, Jiménez and Novo (2016) for an application to vector optimization problems.

**Definition 7.** The Approximate Karush-Kuhn-Tucker (AKKT) conditions is said to hold at $x^0 \in K$ for problem $(P)$ if there are sequences $\{x^k\} \subset \mathbb{R}^n$, $\{u^k\} \subset \mathbb{R}^m_+$ and $\{w^k\} \subset \mathbb{R}^p$, $\{x^k\}$ not necessarily feasible, such that $x^k \longrightarrow x^0$,

$$\lim_{k \longrightarrow \infty} \nabla f(x^k) + \sum_{i \in I(x^0)} u_i^k \nabla g_i(x^k) + \sum_{j=1}^{p} w_j^k \nabla h_j(x^k) = 0$$

and

$$u_i^k = 0 \text{ for } i \notin I(x^0).$$

These asymptotic optimality conditions are genuine first-order necessary optimality conditions, independently of the fulfillment of a contraint qualification and they imply the classical Karush-Kuhn-Tucker conditions under weak constraint qualifications (see Andreani, Haeser and Martinez (2011)). Moreover, they have interesting properties, from an algorithmic point of view.

**Definition 8.** (Andreani, Haeser, Ramos and Silva (2017)). We say that the point $x^0 \in K$ is an *Approximate Stationary Second-Order Point* (AKKT2) for problem $(P)$ if there are sequences $\{x^k\} \subset \mathbb{R}^n$, $\{u^k\} \subset \mathbb{R}^m_+$, $\{w^k\} \subset \mathbb{R}^p$, $\{\eta^k\} \subset \mathbb{R}^p$, $\{\theta^k\} \subset \mathbb{R}^m_+$, $\{\delta_k\} \subset \mathbb{R}_+$, with $u_i^k = 0$ for $i \notin I(x^0)$, $\theta_i^k = 0$ for $i \notin I(x^0)$, such that $x^k \longrightarrow x^0$, $\delta_k \longrightarrow 0$,

$$\lim_{k \longrightarrow \infty} \nabla f(x^k) + \sum_{i \in I(x^0)} u_i^k \nabla g_i(x^k) + \sum_{j=1}^{p} w_j^k \nabla h_j(x^k) = 0$$

and

$$\nabla_x^2 \mathcal{L}(x^k, u^k, w^k) + \sum_{i \in I(x^0)} \theta_i^k \nabla g_i(x^k) \cdot \nabla g_i(x^k)^\top + \sum_{j=1}^{p} \eta_j^k \nabla h_j(x^k) \cdot \nabla h_j(x^k)^\top + \delta_k I$$

is positive semidefinite for $k$ sufficiently large ($I$ is the identity matrix of order $n$).

The quoted authors prove that (AKKT2) is indeed a necessary optimality condition for problem $(P)$.

**Theorem 30.** If $x^0$ is a local optimal solution of $(P)$, then $x^0$ satisfies the (AKKT2) conditions.

Another interesting result of Andreani, Haeser, Ramos and Silva (2017) is that under appropriate first-order constraint qualifications, the (AKKT2) conditions imply the semi-strong second-order necessary conditions of Corollary 1.

**Theorem 31.** Let $x^0$ be a local solution of $(P)$ such that the (AKKT2) conditions hold. If the constraint qualifications (MFCQ) and (WCRC) both hold at $x^0$, then, besides the (KKT) conditions, at $x^0$ it holds

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z \geqq 0, \quad \forall z \in Z_1(x^0).$$

**E)** Finally, we briefly take into consideration a contribute of Ivanov (2015) to second-order necessary optimality conditions (the paper of Ivanov really treats a vector optimization problem). This author considers an optimization problem with no equality constraints: $P = \varnothing$.

$$(P_0): \qquad \begin{cases} \min f(x) \\ \text{subject to } g_i(x) \leqq 0, \ i = 1, ..., m. \end{cases}$$

A direction $d \in \mathbb{R}^n$ is called *critical* at the point $x^0 \in K_0$, where

$$K_0 = \{x \in \mathbb{R}^n : g_i(x) \leqq 0, \quad i = 1, ..., m\},$$

if

$$\nabla f(x^0) d = 0; \quad \nabla g_i(x^0) d \leqq 0, \ i \in I(x^0).$$

For every feasible points $x^0$ and directions $d$ let us introduce the following sets:

$$K(x^0, d) = \left\{ i \in I(x^0) : \nabla g_i(x^0) d = 0 \right\},$$

$$A(x^0, d) = \left\{ z \in \mathbb{R}^n : \forall i \in K(x^0, d) \ \exists \delta_i > 0 \text{ with } g_i(x^0 + td + \frac{1}{2} t^2 z) \leqq 0, \ \forall t \in (0, \delta_i), \right\}$$

$$B(x^0, d) = \left\{ z \in \mathbb{R}^n : \nabla g_i(x^0) z + z^\top \nabla^2 g_i(x^0) z \leqq 0, \ \forall i \in K(x^0, d) \right\}.$$

By definition $A(x^0, d) = B(x^0, d) = \mathbb{R}^n$ if $K(x^0, d) = \varnothing$. If $x^0 \in K_0$, then the set $B(x^0, d)$ is closed but $A(x^0, d)$ is not. Then, following Ivanov, we introduce the *Zangwill Second-Order CQ* at $x^0 \in K_0$ :

$$cl(A(x^0, d)) = B(x^0, d).$$

**Theorem 32 (Ivanov).** Let $x^0 \in K_0$ be a local solution of $(P_0)$ and let $d \neq 0$ be a critical direction. Suppose that the Zangwill Second-Order CQ is satisfied and that the Abadie first-order CQ holds at $x^0$ (i. e. $L(x^0) = T(K_0; x^0)$). Then there exists a Karush-Kuhn-Tucker multipliers vector $u \geqq 0$ such that

$$d^\top \nabla_x^2 \mathcal{L}(x^0, u) d \geqq 0, \ \forall d \in C(x^0).$$

The thesis of Theorem 32 holds also if, instead of the Abadie CQ, we assume the more general Gould-Tolle-Guignard (first-order) CQ:

$$(L(x^0))^* = (T(K_0; x^0))^*$$

i. e., equivalently,

$$L(x^0) = cl(conv(T(K_0; x^0))).$$

# 4. Sufficient Second-Order Optimality Conditions

In differentiable optimization a standard procedure to obtain sufficient second-order optimality conditions from necessary ones, is to replace weak inequalities with strict inequalities. Let us first consider a minimization problem with a set constraint or abstract constraint:

$$(P_1): \qquad \min f(x), \quad x \in S,$$

where $S$ is any subset of $\mathbb{R}^n$ (and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is twice-continuously differentiable on $S$). A classical result, due to Hestenes (1966), is the following one (see also Hestenes (1975)).

**Theorem 33.** Let $x^0 \in S$ and let $\nabla f(x^0) = 0$; if

$$y^\top \nabla^2 f(x^0) y > 0, \quad \forall y \in T(S; x^0) \diagdown \{0\},$$

then $x^0$ is a strict local minimizer for $(P_1)$.

**Proof.** Let us absurdly suppose that $x^0 \in S$ is not a strict local minimizer of $f$ on $S$. Then, there exists for each index $k \in \mathbb{N}$ a point $x^k \in S \cap U_{\frac{1}{k}}(x^0)$ with $x^k \neq x^0$ and $f(x^k) \leqq f(x^0)$. The feasible sequence $\{x^k\}$ converges to $x^0$ and contains a tangentially convergent subsequence We can simply accept that the sequence itself is tangentially convergent to $x^0 : x^k \xrightarrow{y} x^0$. Then it holds $y \in T(S; x^0) \diagdown \{0\}$, but the quotients

$$\frac{f(x^k) - f(x^0)}{\|x^k - x^0\|^2} = \frac{\frac{1}{2}(x^k - x^0)^\top \nabla^2 f(x^0)(x^k - x^0) + o(\|x^k - x^0\|^2)}{\|x^k - x^0\|^2}$$

converge to $\frac{1}{2} y^\top \nabla^2 f(x^0) y \leqq 0$, contrary to what stated in the theorem. □

Hestenes (1966, 1975) shows that the thesis of Theorem 33 can be refined, in the sense that $x^0$ is a strict local minimum point of order 2: there exists a constant $k > 0$ and a neighborhood $N(x^0)$ of $x^0$ such that

$$f(x) \geqq f(x^0) + k \|x^k - x^0\|^2,$$

for all $x \in N(x^0) \cap S$. It must be remarked that the condition

$$\nabla f(x^0) y > 0, \ \forall y \in T(S; x^0) \diagdown \{0\}$$

is a sufficient first-order condition for $x^0 \in S$ to be a strict local solution of $(P_1)$; see, e. g., Hestenes (1975), Giorgi and Guerraggio (1992). More precisely, the above condition assures that $x^0$ is a strict local minimum point of order one of $(P_1)$ : there exists a neighborhood $N(x^0)$ and a constant $k > 0$ such that

$$f(x) \geqq f(x^0) + k \|x - x^0\|, \ \forall N(x^0) \cap S.$$

Let us now consider problem $(P)$. The classical second-order sufficient optimality conditions for $(P)$ are essentially due to McCormick (1967, 1976, 1983) and to Fiacco and McCormick (1968). Previous results are due to Pennisi (1953) and to Pallu de la Barrière (1963).

**Theorem 34.** Let $x^0 \in K$ be a point which verifies the Karush-Kuhn-Tucker conditions:

$$\begin{aligned} &\nabla_x \mathcal{L}(x^0, u, w) = 0; \\ &u_i g_i(x^0) = 0,\ i = 1, ..., m \\ &u_i \geqq 0,\ i = 1, ..., m. \end{aligned}$$

If, for all directions $z \neq 0$, $z \in C(x^0) = Z(x^0)$,

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z > 0,$$

then $x^0$ is a strict local solution of $(P)$.

**Remark 8.**

1) Also for Theorem 34 it is possible to conclude (under its conditions) that really $x^0$ is a strict local minimum point *of order 2* (see Hestenes (1966), (1975)).

2) If at $x^0 \in K$ the *Strict Complementarity Slackness Conditions* hold (i. e. in the Karush-Kuhn-Tucker conditions we have $u_i > 0$, $\forall i \in I(x^0)$), then $Z(x^0) = Z_1(x^0)$, where

$$Z_1(x^0) = \left\{ z \in \mathbb{R}^n : \nabla g_i(x^0) z = 0,\ i \in I(x^0);\ \nabla h_j(x^0) z = 0,\ j = 1, ..., p \right\}$$

is the critical subspace at $x^0$. In such a case the classical results on the sign of a quadratic form subject to a system of homogeneous linear equations may be useful. See, e. g., Chabrillac and Crouzeix (1984), Debreu (1952), Giorgi (2017).

It must be noted that some authors (El-Hodiri (1971, 1991), Castagnoli and Peccati (1979), Takayama (1985)) give the thesis of Theorem 34 in terms of the critical subspace $Z_1(x^0)$, but *without assuming* $u_i > 0$, $\forall i \in I(x^0)$ in the Karush-Kuhn-Tucker conditions. In this case we *do not* obtain sufficient second-order optimality conditions for $(P)$. See Giorgi (2003) and Horsley and Wrobel (2003, 2004) for counterexamples. Unfortunately also the good reference book by Simon and Blume (1994) contains the same error (not present, however, in the Italian version, edited by A. Zaffaroni).

3) Contrary to the case of unconstrained extremum problems, Theorem 34 does not allow to conclude that $x^0$ is also an *isolated* (i. e. locally unique) minimum point, as stated by Fiacco and McCormick (1968) and by McCormick (1976, 1983). The following counterexample is due to Robinson (1982), perhaps the first author to point out such a question. See also Fiacco (1983), Fiacco and Kyparisis (1985) and Still and Streng (1996). Consider the problem

$$\min \frac{x^2}{2},$$

$x \in \mathbb{R}$, subject to $h(x) = 0$, being $h(x) = x^6 \sin\left(\frac{1}{x}\right)$, if $x \neq 0$, $h(x) = 0$, if $x = 0$. The assumptions and conclusions of Theorem 34 hold at $x_0 = 0$ : this point is a strict global minimum point for the problem, however each point $x_k = \frac{1}{k\pi}$, $k$ integer, $k \neq 0$, is a local minimum point for the problem and therefore in each neighborhood of $x_0 = 0$ there are points $x_k$, for $|k|$ sufficiently large. Therefore $x_0 = 0$ is *not* an isolated minimum point for the said problem. See also point c) of the next section.

4) Similarly for necessary second-order optimality conditions, also for sufficient second-order optimality conditions we may require that the multipliers are fixed (as in the previous theorem) or that they are non fixed and speak, respectively, of *strong and weak* sufficient second-order optimality conditions. For example (see, e. g., Ben-Tal (1980) or Still and Streng (1996)), we have the following result.

**Theorem 35.** If $x^0 \in K$ satisfies the Karush-Kuhn-Tucker conditions and for each direction $z \neq 0$, $z \in C(x^0)$, there exists a multiplier vector $(u, w) \in \Lambda(x^0)$ such that

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z > 0, \tag{3}$$

then $x^0$ is a strict local minimum point for $(P)$ of order 2:

$$f(x) \geqq f(x^0) + k \left\| x - x^0 \right\|^2, \quad \forall x \in N(x^0) \cap K. \tag{4}$$

In particular, $x^0$ is a strict local minimum point for $(P)$.

5) Under the Mangasarian-Fromovitz constraint qualification, it is possible to get the converse of Theorem 35 and, therefore, in one sense, to close the gap between necessary and sufficient second-order optimality conditions. See, e. g., Still and Streng (1996).

**Theorem 36.** If $x^0 \in K$ satisfies the (MFCQ) conditions and (4) holds, then condition (3) is satisfied.

Still and Streng (1996) prove the above result under the more general Ben-Tal second-order constraint qualification (or Mangasarian-Fromovitz second-order constraint qualification).

Other (strong) sufficient second-order conditions (sharper than the ones of Theorem 34) are obtained by Hestenes (1966). See also Giorgi and Guerraggio (1996).

**Theorem 37.** Let $x^0 \in K$ and let $(x^0, u, w)$ be the triplet of vectors satisfying the Karush-Kuhn-Tucker conditions. If

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z > 0,$$

for all $z \neq 0$ such that

$$z \in T(K; x^0) \cap \left\{ z \in \mathbb{R}^n : \nabla g_i(x^0) z = 0, \ \forall i \in I_+(x^0, z) \right\},$$

where $I_+(x^0, z) = \{ i \in I(x^0) : u_i > 0 \}$, then $x^0$ is a strict local minimum point of order 2 for problem $(P)$.

Sufficient second-order conditions of the Fritz John-type, i. e. by means of the *Fritz John-Lagrange function* associated to $(P)$,

$$\mathcal{L}_1(x, u_0, u, w) = u_0 f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} w_j h_j(x),$$

have been obtained by Ben-Tal (1980), Bonnans and Shapiro (2000), Han and Mangasarian (1979), Hettich and Jongen (1977), Still and Streng (1996). We recall the definition of the *extended critical cone* $C_1(x^0)$ :

$$C_1(x^0) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \nabla f(x^0)z \leqq 0; \\ \nabla g_i(x^0)z \leqq 0, \quad i \in I(x^0); \\ \nabla h_j(x^0)z = 0, \quad j = 1, ..., p \end{array} \right\}.$$

**Theorem 38 (Han and Mangasarian).** Let $x^0 \in K$ and let the (FJ) conditions be satisfied at $x^0$. If for any $z \neq 0$, $z \in C_1(x^0)$, it holds

$$z^\top \nabla_x^2 \mathcal{L}_1(x^0, u_0, u, w)z > 0,$$

then $x^0$ is a strict local minimum point for $(P)$.

To be more precise, under the conditions of Theorem 38, $x^0$ satisfiesd the "quadratic growth condition". Ben-Tal (1980), Bonnans and Shapiro (2000) and Still and Streng (1996) prove the previous result in its weak form.

**Theorem 39 (Bonnans and Shapiro).** Let $x^0 \in K$; if for each $z \in C_1(x^0)$, $z \neq 0$, there exist multipliers $(u_0, u, w)$ satisfying the Fritz John conditions (i. e. $(u_0, u, w) \in \Lambda_0(x^0)$) such that

$$z^\top \nabla_x^2 \mathcal{L}_1(x^0, u_0, u, w)z > 0,$$

then $x^0$ is a strict local minimum point of order 2 for problem $(P)$.

**Remark 9.** Obviously, if $\Lambda(x^0) \neq \varnothing$, then Theorem 39 is rewritten in the form: for any $z \in C_1(x^0)$, $z \neq 0$, there exist multipliers $(u, w) \in \Lambda(x^0)$ such that

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w)z > 0.$$

These conditions can also be written in the following equivalent form:

$$\sup_{(u,w) \in \Lambda(x^0)} z^\top \nabla_x^2 \mathcal{L}(x^0, u, w)z > 0, \quad \forall z \in C_1(x^0) \setminus \{0\},$$

where the above supremum can be $+\infty$. If the (MFCQ) holds at $x^0$, then $\Lambda(x^0)$ is bounded (closed and convex) and therefore in this case it is possible to substitute "sup" with "max".

Second-order sufficient optimality conditions for local minima that are not necessarily strict local minima were first obtained by Fiacco (1968) using neighborhood information rather than the local information about the point in question. See also Fiacco and McCormick (1968).

**Theorem 40 (Fiacco).** Define

$$S(x^0, \varepsilon, \delta) = \left\{ \begin{array}{l} z \in \mathbb{R}^n : \|z - z'\| \leqq \varepsilon, \text{ for some } z' \in Z(x^0), \ x^0 + \delta_z \in K \\ \text{for some } \delta_z \in (0, \delta), \|z\| = 1, \varepsilon > 0 \end{array} \right\}.$$

If $x^0 \in K$ satisfies the Karush-Kuhn-Tucker conditions with multipliers vectors $(u, w)$ and there exists a $\bar{S} = S(x^0, \bar{\varepsilon}, \bar{\delta})$ such that for all $z \in \bar{S}$ we have

$$z^\top \nabla_x^2 \mathcal{L}(x^0 + \lambda\delta_z z, u, w)z \geqq 0,$$

for all $\lambda \in (0, 1)$, then $x^0$ is a local minimum of problem $(P)$.

This result has been generalized by Liu and Liu (1995).

Since now we have discussed sufficient second-order conditions for local (strict or weak) optimality. What to say about sufficient second-order conditions for global optimality? We know that in general, in order to obtain global optimality of a feasible point $x^0$ for $(P)$, some convexity or generalized convexity assumptions on the functions involved in $(P)$ must be imposed. But if these functions are $\mathcal{C}^2$, there are chracterizations of convexity and generalized convexity in terms of second-order derivatives. See, e. g., Cambini and Martein (2009). The second-order global sufficient conditions can thus be obtained, even if they are quite cumbersome to be described. A simpler sufficient second-order global optimality condition is contained in the following result, due to Bomze (2016).

**Theorem 41.** Suppose that $K$ is convex; if $x^0 \in K$ satisfies the Karush-Kuhn-Tucker conditions for $(P)$ with multipliers vectors $(u, w)$, if

$$z^\top \nabla_x^2 \mathcal{L}(x, u, w)z \geqq 0$$

for all $z$ such that $z \in L(x^0)$, i. e.

$$\begin{cases} \nabla g_i(x^0)z \leqq 0, & \forall i \in I(x^0) \\ \nabla h_j(x^0)z = 0, & j = 1, ..., p, \end{cases}$$

and *for all* $x \in K$, then $x^0$ is a global solution of problem $(P)$.

We note that $K$ is convex if, for example, all functions $g_i(x)$ are quasiconvex and all functions $h_j$ are linear affine. However, the previous result requires checking the sign of a constrained quadratic form for all feasible vectors $x$, which may be not easy, unless the said quadratic form (the Hessian of the Lagrangian function) does not depend from $x$. This is the case, e. g., of a quadratic objective function subject to quadratic and/or linear affine functions.

# 5. Other Approaches to Sufficient Second-Order Optimality Conditions

**A)** Hestenes (1975) obtains necessary and sufficient optimality conditions for $(P)$ and for problem $(P_1)$ (i. e. for the minimization problem with an abstract constraint), by introducing the notion of *support functions.* Let us consider problem $(P_1)$ :

$$(P_1) : \qquad \min f(x), \quad x \in S \subset \mathbb{R}^n.$$

Let us consider $x^0 \in S$ and a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, twice continuously differentiable on $S$ and satisfying the relations

$$F(x) \leqq f(x) \text{ on } S; \quad F(x^0) = f(x^0), \quad \nabla F(x^0) = 0.$$

Such a function $F$ is called by Hestenes *lower support function for* $f$ at $x^0$.

If we consider problem $(P)$ it is clear that the function

$$F = f + \sum u_i g_i + \sum w_j h_j$$

is a lower support function of this type at $x^0 \in K$.

Lower support functions are associated with minima. The corresponding functions $F$ for a maximum point $x^0$ of $f$ on $S$ are called by Hestenes *upper support functions;* they are defined by the relations

$$F(x) \geqq f(x) \text{ on } S; \quad F(x^0) = f(x^0); \quad \nabla F(x^0) = 0.$$

These concepts and the related optimality results of Hestenes have been generalized to multiobjective programming problems by Jiménez and Novo (2002).

**Theorem 42 (Hestenes).** Let $x^0 \in S$ and suppose that

$$\nabla f(x^0)v = 0 \tag{5}$$

for some $v \in T(S; x^0)$. Suppose further that for all $v \neq 0$ satisfying relation (5) there exists a lower support function $F$ such that

$$v^\top \nabla^2 F(x^0)v > 0.$$

Then, there exist a neighborhood $N(x^0)$ and a constant $k > 0$ such that

$$f(x) \geqq f(x^0) + k \left\| x - x^0 \right\|^2, \quad \forall x \in N(x^0) \cap S.$$

It must be observed that it is not required $F$ to be the same for every $v \in T(S; x^0)$ satisfying (5). In other words, $F$ can be chosen to be independent of $v$.

For what regards problem $(P)$ it must be remarked that if the Karush-Kuhn-Tucker conditions hold at $x^0 \in K$, the Lagrangian function $F = f + ug + wh$ is a lower support function for $(P)$ at $x^0 \in K$, i. e.

$$F(x) \leqq f(x) \text{ on } K; \quad F(x^0) = f(x^0); \quad \nabla F(x^0) = 0.$$

The next result can be obtained as a consequence of the previous one.

**Theorem 43.** Let us consider problem $(P)$ and its feasible set $K$. Let $x^0 \in K$; if for those vectors $v \in T(K; x^0)$, $v \neq 0$ and such that $\nabla f(x^0)v = 0$, there is a lower support function $F = \mathcal{L}(x, u, w)$, with $u \geqq 0$ ($u_i = 0$ for $i \notin I(x^0)$) and $\nabla F(x^0) = 0$ and such that

$$v^\top \nabla^2 F(x^0)v > 0,$$

then $x^0$ is a strict local minimum of order 2 for $(P)$.

**B)** Second-order tangent sets and second-order tangent cones have been used by various authors (e. g. Cambini and Martein (2003), Cambini, Martein and Vlach (1999), Penot (1994, 1998, 2000), Jiménez and Novo (2004)) to obtain sufficient second-order optimality conditions for $(P)$ or for $(P_1)$. With reference to $(P_1)$ we have the following results.

**Theorem 44.** Let us consider $(P_1)$ and let $x^0 \in S$. If for each $v \in T(S; x^0) \diagdown \{0\}$ such that $\nabla f(x^0)v = 0$ the following systems in $w \in \mathbb{R}^n$ are incompatible

$$(a) \qquad \begin{cases} w \in T^2(S, x^0, v) \cap v^{\perp}, \\ \nabla f(x^0)w + v^{\top} \nabla^2 f(x^0) v \leqq 0; \end{cases}$$

$$(b) \qquad \begin{cases} w \in T''(S, x^0, v) \cap v^{\perp} \diagdown \{0\}, \\ \nabla f(x^0)w \leqq 0, \end{cases}$$

then $x^0$ is a strict local minimum point of order 2 for $(P_1)$.

This result is a reformulation for the scalar case of a result of Jiménez and Novo (2004, Corollary 4.10) given for Pareto problems. It improves corresponding results of Cambini, Martein and Vlach (1999) and of Penot (1998, 2000). For second-order sufficient conditions in terms of $T''(K, x^0, v)$ or $T^2(K, x^0, v)$ the reader may consult Penot (1998, 2000) or Gutierrez, Jiménez and Novo (2010).

**C)** We have already remarked that Robinson (1982) provided a numerical example where, contrary to the unconstrained case, the classical sufficient second-order optimality conditions of Theorem 34 *do not assure* that $x^0$ is an isolated (i. e. locally unique) local minimizer for $(P)$. See also Fiacco (1983). Conditions sufficient for $x^0$ to be an isolated local minimum are obtained by Robinson (1982) by strenghtening the assumptions of Theorem 34 in two ways.

**Theorem 45 (Robinson).** Suppose that the Karush-Kuhn-Tucker conditions hold at $x^0 \in K$ for $(P)$ with multipliers vectors $(u, w)$ and that the Mangasarian-Fromovitz constraint qualification holds at $x^0$. Moreover, assume that the following *General Second-Order Sufficient Conditions* hold at $x^0$ :

- The Strong Second-Order Sufficient Conditions of Theorem 34 hold at $x^0$ with $(u, w)$ for *every* $(u, w) \in \Lambda(x^0)$.

Then $x^0$ is an isolated local minimum point of $(P)$, i. e. there exists a neighborhood $N(x^0)$ of $x^0$ such that $x^0$ is the only local minimum point of $(P)$ in $N(x^0)$.

Note that if the linear independence (LICQ) or also the *Strict Mangasarian-Fromovitz CQ* are substituted for (MFCQ) in Theorem 45, then $\Lambda(x^0)$ becomes a singleton and in this case Theorem 34 assures that $x^0$ is an isolated local minimum point. The same is true for the classical second-order sufficient conditions in the unconstrained case. The results of Robinson have been generalized by Still and Streng (1996), by Kyparisis and Fiacco (1986) in an unpublished paper, by Liu (1992) and by Liu and Liu (1995). For the reader's convenience we report the results of Still and Streng (1996).

**Theorem 46.** Let $x^0 \in K$ and suppose that either $(i)$ or $(ii)$ holds, where

$(i)$ It holds $C_1(x^0) = \{0\}$ and the Mangasarian-Fromovitz CQ is fulfilled at $x^0$.

$(ii)$ It holds $C_1(x^0) \neq \{0\}$ , $\Lambda_0(x^0) \neq 0$, the Second-Order Mangasarian-Fromovitz CQ (or Ben-Tal Second-Order CQ) is satisfied at $x^0$ and for any nonzero vector $z \in C_1(x^0)$ and any nonzero multipliers $(u_0, u, w) \in \Lambda_0(x^0)$ we have

$$z^\top \nabla_x^2 \mathcal{L}_1(x^0, u_0, u, w) z > 0.$$

Then, $x^0$ is an isolated strict local minimum of order one in case $(i)$, of order two in case $(ii)$.

# 6. Applications of Second-Order Optimality Conditions to Sensitivity Analysis for a Parametric Nonlinear Programming Problem

A typical application of second-order optimality conditions is in sensitivity analysis of a nonlinear programming problem. The basic sensitivity results in nonlinear programming problems under second-order conditions are due to Fiacco (1976, 1980, 1983). Here we follow mainly the survey paper of Fiacco and Kyparisis (1985); see also Giorgi and Zuccotti (2018).

A general parametric nonlinear programming problem is defined as:

$$(P(\varepsilon)) : \quad \begin{cases} \min f(x,\varepsilon) & \\ \text{subject to:} & g_i(x,\varepsilon) \leqq 0, \ i = 1, ..., m, \\ & h_j(x,\varepsilon) = 0, \ j = 1, ..., p, \end{cases}$$

where $\varepsilon \in \mathbb{R}^r$ is a perturbation parameters vector, $f, g_i, h_j : \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}$; it is assumed that all these functions are $\mathcal{C}^2$ in $(x, \varepsilon)$ at least in a neighborhood of a feasible point $(x^0, \varepsilon^0)$. The Lagrangian function associated to $(P(\varepsilon))$ is

$$\mathcal{L}(x, u, w, \varepsilon) = f(x,\varepsilon) + \sum_{i=1}^m u_i g_i(x,\varepsilon) + \sum_{j=1}^p w_j h_j(x,\varepsilon).$$

The following theorem, proved by Fiacco (1976), is a basic sensitivity result for $(P(\varepsilon))$ under second-order differentiability assumptions.

**Theorem 47 (Fiacco).** Suppose that the strong second-order sufficient conditions of Theorem 34 hold for $(P(\varepsilon^0))$ at $x^0$ with associated Karush-Kuhn-Tucker multipliers $(u, w)$, that the linear independence (LICQ) holds at $x^0$ for $(P(\varepsilon^0))$ and that the *Strict Complementarity Slackness Condition* holds at $x^0$ with respect to $u$ for $(P(\varepsilon^0))$, i. e. $u_i > 0$ when $g_i(x^0, \varepsilon^0) = 0$, $i \in I(x^0)$, where

$$I(x^0) = \left\{ i : g_i(x^0, \varepsilon^0) = 0 \right\}.$$

Then:

$(a)$ The point $x^0$ is an isolated local minimum point of $(P(\varepsilon^0))$ and the associated Karush-Kuhn-Tucker multipliers $u$ and $w$ are unique.

$(b)$ For $\varepsilon$ in a neighborhood of $\varepsilon^0$, there exists a unique vector function

$$y(\varepsilon) = [x(\varepsilon), u(\varepsilon), w(\varepsilon)]^\top$$

satisfying the strong second-order sufficient conditions of Theorem 34 for a local minimum of $(P(\varepsilon))$ such that $y(\varepsilon^0) = [x^0, u, w]^\top$ and, hence, $x(\varepsilon)$ is a locally unique local minimizer of $(P(\varepsilon))$ with associated Karush-Kuhn-Tucker multipliers vectors $u(\varepsilon)$ and $w(\varepsilon)$.

$(c)$ The linear independence (LICQ) and Strict Complementarity Slackness Condition hold at $x(\varepsilon)$ for $\varepsilon$ near $\varepsilon^0$.

Fiacco (1976) also shows that the derivative of $y(\varepsilon)$ can be calculated by noting that the following system of Karush-Kuhn-Tucker equations will hold at $y(\varepsilon)$ for $\varepsilon$ near $\varepsilon^0$ under the assumptions of Theorem 47,

$$\begin{cases} \nabla_x \mathcal{L}\,[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon] = 0, \\ u_i(\varepsilon) g_i(x(\varepsilon), \varepsilon) = 0, \ i = 1, ..., m, \\ h_j(x(\varepsilon), \varepsilon) = 0, \ j = 1, ..., p. \end{cases} \tag{6}$$

Since these assumptions imply that the Jacobian, $M(\varepsilon)$, of system (6) with respect to $(x, u, w)$ is nonsingular, one obtains

$$M(\varepsilon) \nabla_\varepsilon y(\varepsilon) = -N(\varepsilon) \tag{7}$$

and

$$\nabla_\varepsilon y(\varepsilon) = -\left[M(\varepsilon)\right]^{-1} N(\varepsilon),$$

where $N(\varepsilon)$ is the Jacobian of system (6) with respect to $\varepsilon$.

System (7) at $\varepsilon = \varepsilon^0$ can be written in the form

$$M^0 \begin{bmatrix} \nabla_\varepsilon x(\varepsilon^0) \\ \nabla_\varepsilon u(\varepsilon^0) \\ \nabla_\varepsilon w(\varepsilon^0) \end{bmatrix} = -N^0$$

where

$$M^0 = \begin{bmatrix} \nabla_x^2 \mathcal{L} & \nabla_x g_1^\top & \cdots & \nabla_x g_m^\top & \nabla_x h_1^\top & \cdots & \nabla_x h_p^\top \\ u_1 \nabla_x g_1 & g_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ u_m \nabla_x g_m & 0 & \cdots & g_m & 0 & \cdots & 0 \\ \nabla h_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \nabla h_p & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{8}$$

and

$$N^0 = \left[\nabla_{\varepsilon x}^2 \mathcal{L}^\top, u_1 \nabla_\varepsilon g_1^\top, ..., u_m \nabla_\varepsilon g_m^\top, \nabla_\varepsilon h_1^\top, ..., \nabla_\varepsilon h_p^\top\right]^\top$$

are evaluated at $(x^0, u, w, \varepsilon^0)$.

The next theorem, due to McCormick (1976), shows that the conditions imposed in Theorem 47 are also essentially necessary, under appropriate regularity assumptions, for the invertibility of the Jacobian matrix $M^0$.

**Theorem 48 (McCormick).** Suppose that the strong second-order necessary conditions of Theorem 6 hold for a local minimum point $x^0$ of $(P(\varepsilon^0))$ with associated Karush-Kuhn-Tucker multipliers $(u, w)$. Then, the Jacobian matrix $M^0$ given by (8) is invertible if and only if the strong second-order sufficient conditions, given by Theorem 34, the linear independence (LICQ) and the Strict Complementarity Slackness Condition hold at $x^0$ with $(u, w)$ for $(P(\varepsilon^0))$.

The previous classical results of Fiacco have been extended towards various directions. For example, Jittorntrum (1984) and Robinson (1980) relaxe the strict complementarity slackness assumption but use a stronger second-order sufficient optimality condition, a condition that these authors call "strong second-order sufficient conditions", but that we call, following Ruszczynski (2006) and in order to avoid confusions, *"semi-strong second-order sufficient conditions":*

$$z^\top \nabla_x^2 \mathcal{L}(x^0, u, w) z > 0,$$

for all $z \neq 0$ such that

$$\begin{aligned} \nabla_x g_i(x^0, \varepsilon^0) z &= 0, \ \forall i \in I^+(x^0, \varepsilon^0); \\ \nabla_x h_j(x^0, \varepsilon^0) z &= 0, \ \forall j = 1, ..., p. \end{aligned}$$

**Theorem 49 (Jittorntrum, Robinson).** Suppose that the Karush-Kuhn-Tucker conditions hold at $x^0$ for $(P(\varepsilon^0))$ with Karush-Kuhn-Tucker multipliers $(u, w)$, that the additional Semi-Strong Second-Order Sufficient Conditions hold at $x^0$ with $(u, w)$ and that the (LICQ) condition holds at $x^0$ for $(P(\varepsilon^0))$. Then:

$(a)$ The point $x^0$ is an isolated local minimum point of $(P(\varepsilon^0))$ and the associated multipliers vectors $u$ and $w$ are unique.

$(b)$ For $\varepsilon$ in a neighborhood of $\varepsilon^0$, there exists a unique continuous vector function

$$y(\varepsilon) = [x(\varepsilon), u(\varepsilon), w(\varepsilon)]^\top$$

satisfying the Karush-Kuhn-Tucker conditions and the semi-strong second-order sufficient conditions for a local minimum of $(P(\varepsilon))$ such that $y(\varepsilon^0) = (x^0, u, w)^\top$ and, hence, $x(\varepsilon)$ is a locally unique local minimum point of $(P(\varepsilon))$ with associated unique multipliers vectors $u(\varepsilon)$ and $w(\varepsilon)$.

$(c)$ There exists $t > 0$ and $d > 0$ such that for all $\varepsilon$ with $\|\varepsilon - \varepsilon^0\| < d$, it follows that

$$\left\| y(\varepsilon) - y(\varepsilon^0) \right\| \leqq t \left\| \varepsilon - \varepsilon^0 \right\|.$$

Another direction of extension of the previous basic results of Fiacco has been considered by Kojima (1980), who substitutes the weaker Mangasarian-Fromovitz constraint qualification for the (LICQ) condition. However, Kojima makes a further strengthening of the second-order optimality conditions. We denote by $\Lambda(x, \varepsilon)$ the set of Karush-Kuhn-Tucker multipliers at $x$ for $(P(\varepsilon))$, i. e.

$$\Lambda(x, \varepsilon) = \left\{(u, w) \in \mathbb{R}_+^m \times \mathbb{R}^p : \nabla_x \mathcal{L}(x, u, w) = 0,\ u_i g_i(x, \varepsilon) = 0,\ u_i \geqq 0,\ i = 1, ..., m\right\}.$$

We say that at $x^0 \in K(\varepsilon^0)$, feasible set of $(P(\varepsilon^0))$, the *General Semi-Strong Second-Order Sufficient Conditions* hold with multipliers vectors $(u, w)$ if:

- The semi-strong second-order sufficient conditions hold at $x^0$ for *every* $(u, w) \in \Lambda(x^0, \varepsilon^0)$.

**Theorem 50 (Kojima (1980), Fiacco and Kyparisis (1985)).** Suppose that the Karush-Kuhn-Tucker conditions hold at $x^0$ for $(P(\varepsilon^0))$ with multipliers vectors $(u, w)$, that the additional General Semi-Strong Second-Order Sufficient Conditions hold at $x^0$ and that the Mangasarian-Fromovitz CQ holds at $x^0$ for $(P(\varepsilon^0))$. Then:

$(a)$ The point $x^0$ is an isolated local minimum point of $x^0$ and the set $\Lambda(x^0, \varepsilon^0)$ is compact and convex.

$(b)$ There are neighborhoods $N(x^0)$ of $x^0$ and $N(\varepsilon^0)$ of $\varepsilon^0$ such that for $\varepsilon$ in $N(\varepsilon^0)$ there exists a unique continuous vector function $x(\varepsilon)$ in $N(x^0)$ satisfying the Karush-Kuhn-Tucker conditions with some $[u(\varepsilon), w(\varepsilon)] \in \Lambda(x(\varepsilon), \varepsilon)$ and the General Semi-strong Second-Order Sufficient Conditions, such that $x(\varepsilon^0) = x^0$, and hence $x(\varepsilon)$ is the locally unique local minimizer of $(P(\varepsilon))$ in $N(x^0)$.

$(c)$ The Mangasarian-Fromovitz CQ holds at $x(\varepsilon)$ for $\varepsilon$ in $N(\varepsilon^0)$.

We note that Gauvin (1977) previously proved, with reference to $(P)$, that the Mangasarian-Fromovitz CQ is a necessary and sufficient condition to have $\Lambda(x^0)$ closed and bounded.

Fiacco and Kyparisis (1985) extend further the previous results by means of an assumption they call *General Strict Complemenmtarity Slackness Condition:*

- The *General Strict Complementarity Slackness Condition* holds at $x^0$ for $(P(\varepsilon^0))$ if the strict complementarity slackness condition holds at $x^0$ with respect to the Karush-Kuhn-Tucker multipliers $(u, w)$ for *every* $(u, w) \in \Lambda(x^0, \varepsilon^0)$.

**Theorem 51 (Fiacco and Kyparisis (1985)).** Suppose that $x^0$ is feasible for $(P(\varepsilon^0))$ and that the Karush-Kuhn-Tucker conditions, the General Strict Complementarity Slackness Condition and the Mangasarian-Fromovitz CQ hold at $x^0$. Then $\Lambda(x^0, \varepsilon^0)$ is a *singleton,* and the (LICQ) condition holds at $x^0$.

The next result, due to Fiacco and Kyparisis (1985), essentially follows as a corollary to Theorem 51.

**Theorem 52 (Fiacco and Kyparisis (1985)).** The following two sets of assumptions are equivalent:

$(a)$ The Karush-Kuhn-Tucker conditions, the second-order strong sufficient conditions of Theorem 34, the strict complementarity slackness condition and the (LICQ) condition hold at the feasible point $x^0$ for $(P(\varepsilon^0))$.

$(b)$ The Karush-Kuhn-Tucker conditions, the general second-order (strong) sufficient conditions of Theorem 45, the general strict complementarity slackness condition and the Mangasarian-Fromovitz CQ hold at the feasible point $x^0$ for $(P(\varepsilon^0))$.

We note that the results of the present section have been carried out under progressively weaker assumptions. However, in absence of inequality constraints, all of these sets of conditions reduce to the Lagrange classical optimality conditions, second-order classical sufficient optimality conditions and regularity condition (i. e. invertibility) on the Jacobian matrix $\nabla h(x^0)$.

# References


B. AGHEZZAF (1999), *Second-order necessary conditions of the Kuhn-Tucker type in multiobjective programming problems*, Control and Cybernetics, 28, 213-224.

B. AGHEZZAF and M. HACHIMID (1998), *Second order necessary and sufficient conditions for efficiency in multiobjective programming problems*, Investigacion Operativa, 6, 19-28.

B. AGHEZZAF and M. HACHIMI (1999), *Second order optimality conditions in multiobjective optimization problems*, J. Optim. Theory Appl., 102, 37-50.

R. ANDREANI, R. BEHLING, G. HAESER and P. J. S. SILVA (2017), *On second-order optimality conditions for nonlinear optimization*, Optim. Methods and Software, 32, 22-38.

R. ANDREANI, E. G. BIRGIN, J. M. MARTINEZ and M. L. SCHUVERDT (2010), *Second-order negative-curvature methods for box-constrained optimization*, Computational Optimization and Applications, 45, 209-236.

R. ANDREANI, C. E. ECHAGÜE and M. L. SCHUVERDT (2010), *Constant-rank condition and second-order constraint qualifications*, J. Optim. Theory Appl., 146, 255-266.

R. ANDREANI, G. HAESER and J. M. MARTINEZ (2011), *On sequential optimality conditions for smooth constrained optimization*, Optimization, 60, 627-641.

R. ANDREANI, G. HAESER, A. RAMOS and P. J. S. SILVA (2017), *A second-order sequential optimality condition associated to the convergence of optimization algorithms*, IMA J. Numer, Anal., 37, 1902-1929. *Erratum* in IMA J. Numer. Anal. 37, N.4, e 1.

R. ANDREANI, J. M. MARTINEZ and M. L. SCHUVERDT (2007), *On second-order optimality conditions for nonlinear programming*, Optimization, 56, 529-542.

M. ANITESCU (2000), *Degenerate nonlinear programming with a quadratic growth condition*, SIAM J. Optim., 10, 1116-1135.

A. V. ARUTYUNOV (1991), *Perturbations of extremum problems with constraints and necessary optimality conditions*, J. Sov. Math., 54, 1342-1400.

A. ARUTYUNOV and F. L. PEREIRA (2006), *Second-order necessary optimality conditions for problems without a priori normality assumptions*, Mathematics of Operations Research, 31, 1-12.

J. P. AUBIN and H. FRANKOWSKA (1990), Set-Valued Analysis, Birkhäuser, Boston.

M. AVRIEL (1976), Nonlinear Programming. Analysis and Methods, Prentice Hall, Englewood Cliffs, N. J.

A. BACCARI (2004), *On the classical necessary second-order optimality conditions*, J. Optim. Theory Appl., 123, 213-221.

A. BACCARI and A. TRAD (2004), *On the classical necessary second-order optimality conditions in the presence of equality and inequality constraints*, SIAM J. Optim., 15, 394-408.

M. S. BAZARAA, J. J. GOODE and M. Z. NASHED (1974) *On the cone of tangents with applications to mathematical programming*, J. Optim. Theory Appl., 13, 389-426.

M. S. BAZARAA and C. M. SHETTY (1976), Foundations of Optimization, Springer Verlag, new York.

M. S. BAZARAA, H. D. SHERALI and C. M. SHETTY (2006), Nonlinear Programming. Theory and Algorithms. Third Edition, Wiley-Interscience, New York.

C. R. BECTOR, S. CHANDRA and J. DUTTA (2005), Principles of Optimization Theory, Alpha Science International Ltd., Harrow, U. K.

R. BEHLING, G. HAESER, A. RAMOS and D. S. VIANA (2017), *On a conjecture in second-order optimality conditions*, Extended Technical Report, arXiv:1706.07833.

R. BEHLING, G. HAESER, A. RAMOS and D. S. VIANA (2018), *On a conjecture in second-order optimality conditions*, J. Optim. Theory Appl.,176, 625-633.

A. BEN-TAL (1980), *Second-order and related extremality conditions in nonlinear programming*, J. Optim. Theory Appl., 31, 143-165.

A. BEN-TAL and J. ZOWE (1982), *A unified theory of first and second order conditions for extremum problems in topological vector spaces*, Mathematical Programming Study, 19, 39-76.

G. BIGI (2006), *On sufficient second order optimality conditions in multiobjective optimization*, Math. Meth. Oper. Res., 63, 77-85.

G. BIGI and M. CASTELLANI (2000), *Second-order optimality conditions for differentiable multiobjective problems*, R.A.I.R.O., 34, 411-426.

G. BIGI and M. CASTELLANI (2004), *Uniqueness of KKT multipliers in multiobjective optimization*, Applied Mathematics Letters, 17, 1285-1290.

S. BOLINTINEANU and M. EL MAGHRI (1998), *Second-order efficiency conditions and sensitivity of efficient points*, J. Optim. Theory Appl., 98, 569-592.

I. M. BOMZE (2016), *Copositivity for second-order optimality conditions in smooth optimization problems*, Optimization, 65, 779-795.

J. F. BONNANS (1989), *A semistrong sufficiency condition for optimality in nonconvex programming and its connection to the perturbation problem*, J. Optim. Theory Appl., 60, 7-18.

J. F. BONNANS and R. COMINETTI (1996), *Perturbed optimization in Banach spaces I, II, III*, SIAM J. Control Optim., 34, 1151-1171; 1172-1189; 1555-1567.

J. F. BONNANS, R. COMINETTI and A. SHAPIRO (1999), *Second-order optimality conditions based on parabolic second-order tangent sets*, SIAM J. Optim., 9, 466-492.

J. F. BONNANS and A. IOFFE (1995), *Second-order sufficiency and quadratic growth for non isolated minima*, Mathematics of Operations Research, 20, 801-817.

J. F. BONNANS and A. SHAPIRO (2000), Perturbation Analysis of Optimization Problems, Springer Verlag, Berlin.

J. BRINKHUIS (2009), *A linear programming proof of the second order conditions of non-linear programming*, European J. of Operational Research, 192, 1001-1007.

E. BURGER (1955), *On extrema with side conditions*, Econometrica, 23, 451-452.

J. V. BURKE and C. M. FERRIS (1993), *Weak sharp minima in mathematical programming*, SIAM J. Control Optim., 31, 1340-1359.

R. CAMBINI (1998), *Second-order optimality conditions in multiobjective programming*, Optimization, 44, 139-160.

A. CAMBINI and L. MARTEIN (2002), *Second order optimality conditions*; in B. K. Dass (Ed.), Discrete Mathematics and Allied Topics, Taru Publications & Academic Forum, New Delhi, 243-252.

A. CAMBINI and L. MARTEIN (2003), *First and second-order optimality conditions in vector optimization*; in A. Cambini and Others (Eds.), Generalized Convexity, Generalized Monotonicity, Optimality Conditions and Duality in Scaler *(sic)* and Vector Optimization, Taru Publications and Academic Forum, New Delhi, 295-319.

A. CAMBINI and L. MARTEIN (2009), Generalized Convexity and Optimization. Theory and Applications, Springer Verlag, Berlin.

A. CAMBINI, L. MARTEIN and M. VLACH (1999), *Second order tangent sets and optimality conditions*, Math. Japonica, 49, 451-461.

E. CASAS and F. TROLTZSCH (2002), *Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory*, SIAM J. Optimization, 13, 406-431.

C. CARATHEODORY (1989), Calculus of Variations and Partial Differential Equations of First Order, Chelsea, New York.

E. CASTAGNOLI and L. PECCATI (1979), Matematica per l'Analisi Economica, ETAS LIBRI, Milano.

M. CASTELLANI (1994), *Variational sets and second-order necessary optimality conditions*, Le Matematiche, XLIX, 189-203.

M. CASTELLANI (1996), *A necessary second-order optimality condition in nonsmooth mathematical programming*, Operations Research Letters, 19, 79-86.

M. CASTELLANI and M. PAPPALARDO (1996), *Local second-order approximations and applications in optimization*, Optimization, 37, 305-321.

M. CASTELLANI and M. PAPPALARDO (1999), *A unifying approach for higher-order optimality conditions*, Communications of Applied Analysis, 3, 15-28.

Y. CHABRIALLAC and J.-P. CROUZEIX (1984), *Definiteness and semidefiniteness od quadratic forms revisited*, Linear Algebra and Its Appl., 63, 283-292.

R. COMINETTI (1990), *Metric regularity, tangent sets, and second-order optimality conditions*, Appl. Math. Optim., 21, 265-287.

E. CONSTANTIN (2006), *Second-order necessary conditions based on second order tangent cones*, Math. Sciences Research J., 10, 42-56.

E. CONSTANTIN (2016), *Second-order necessary conditions for set constrained nonsmooth optimization problems via second-order projective tangent cones*, Libertas Mathematica (new series), 36, 73-95.

L. CROMME (1978), *Strong uniqueness: a far reaching criterion for the convergence of iterative procedures*, Numerische Mathematik, 29, 179-193.

M. DALDOUL and A. BACCARI (2009), *An application of matrix computations to classical second-order optimality conditions*, Optim. Lett., 3, 547-557.

G. DEBREU (1952), *Definite and semidefinite quadratic forms*, Econometrica, 20, 295-300.

S. DI (1996), *Classical optimality conditions under weaker assumptions*, SIAM J. Optimization, 6, 178-197.

M. A. EL-HODIRI (1971), Constrained Extrema: Introduction to the Differentiable Case with Economic Applications, Lecture Notes in Operations Research and Mathematical Systems, vo. 56, Springer, Berlin-Heidelberg-New York.

M. A. EL-HODIRI (1991), Extrema of Smooth Functions. With Examples from Economic Theory, Springer Verlag, Berlin.

M. C. FERRIS and O. L. MANGASARIAN (1992), *Minimum principle sufficiency*, Math. Programming, 57, 1-14.

A. V. FIACCO (1968), *Second order sufficient conditions for weak and strict constrained extrema*, SIAM Journal on Applied Mathematics, 16, 105-108.

A. V. FIACCO (1976), *Sensitivity analysis for nonlinear programming using penalty methods*, Math. Programming, 10, 287-311.

A. V. FIACCO (1980), *Nonlinear programming sensitivity analysis results using strong second order assumptions*; in L. C. W. Dixon and G. P. Szegö (eds.), Numerical Optimization and Dynamical Systems, North Holland, Amsterdam, 327-348.

A. V. FIACCO (1983), Introduction to Sensitivity and Stability Analysis in Nonlinear Programming, Academic Press, New York.

A. V. FIACCO and J. KYPARISIS (1985), *Sensitivity analysis in nonlinear programming under second-order assumptions*; in A. Bagchi and H. Th. Jongen (Eds.), Systems and Optimization, Proceedings of the Twente Workshop, Enschede, The Netherlands, April 16-18, 1984, Springer Verlag, Berlin, 74-97.

A. V. FIACCO and G. P. McCORMICK (1968a), Nonlinear Programming: Sequential Unconstrained Minimization Techniques, J. Wiley, New York.

A. V. FIACCO and G. P. McCORMICK (1968b), *Asymptotic conditions for constrained minimization*, RAC-TP-340, Research Analysis Corporation, Mc Lean, Virginia.

R. FLETCHER (1987), Practical Methods of Optimization (second edition), John Wiley & Sons, New York.

W. FORST and D. HOFFMANN (2010), Optimization Theory and Practice, Springer, New York.

R. FRISCH (1966), Maxima and Minima: Theory and Economic Applications, D. Reidel, Dordrecht.

J. GAUVIN (1977), *A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming*, Math. Programming, 12, 136-138.

H. GFRERER (2007), *Second order necessary conditions for nonlinear optimization with abstract constraints: the degenerate case*, SIAM J. Optimization, 18, 589-612.

G. GIORGI (2003), *Some remarks on second-order optimality conditions*, International Journal of Mathematical Sciences, 2, 97-102.

G. GIORGI (2017), *Various proofs of the Sylvester criterion for quadratic forms*, Journal of Mathematics Research, 9, 55-66.

G. GIORGI (2018), *A guided tour in constraint qualifications for nonlinear programming under differentiability assumptions*, DEM Working Paper Series N. 160, Department of Economics and Management, University of Pavia, Pavia (Italy). economiaweb.unipv.it

G. GIORGI (2019), *Notes on constraint qualifications for second-order optimality conditions,* in preparation.

G. GIORGI and A. GUERRAGGIO (1992), *On the notion of tangent cone in mathematical programming*, Optimization, 25, 11-23.

G. GIORGI and A. GUERRAGGIO (1996), *Uno schema per un approccio unificante a condizioni di ottimalità del primo e del secondo ordine*, Atti del XX Convegno A.M.A.S.E.S., Urbino, 5-7 Settembre 1996, Università degli Studi di Urbino, 281-304.

G. GIORGI and A. GUERRAGGIO (2002), *Characterizations, comparisons, algebraic and topological properties of tangent cones*, Journal of Statistics & Management Systems, 5, 275-294.

G. GIORGI, A. GUERRAGGIO and J. THIERFELDER (2004), Mathematics of Optimization: Smooth and Nonsmooth Case, Elsevier, Amsterdam.

G. GIORGI, B. JIMENEZ and V. NOVO (2010), *An overview of second-order tangent sets and their application to vector optimization*, Bol. Soc. Esp. Mat. Apl., 52, 73-96.

G. GIORGI, B. JIMENEZ and V. NOVO (2016), *Approximate Karush-Kuhn-Tucker condition in multiobjective optimization*, J. Optim. Theory Appl., 171, 70-89.

G. GIORGI and T. H. KJELDSEN (2014), Traces and Emergence in Nonlinear Programming, Birkhäuser, Basel.

G. GIORGI and C. ZUCCOTTI (2008), *Osservazioni sulle condizioni di ottimalità del secondo ordine in programmazione matematica*, Quaderni di Ricerca - X - Facoltà di Economia, Dipartimento di Ricerche Aziendali, Univ. di Pavia, Aracne Editrice, Roma.

G. GIORGI and C. ZUCCOTTI (2018), *A tutorial on sensitivity and stability in nonlinear programming and variational inequalities under differentiability assumptions*, DEM Working Paper Series N. 159, Department of Economics and Management, University of Pavia, Pavia (Italy). economiaweb.unipv.it

F. J. GOULD and J. W. TOLLE (1971), *A necessary and sufficient qualification for constrained optimization*, SIAM Journal on Applied Mathematics, 20, 164-172.

A. GUERRAGGIO, D. T. LUC and N. B. MINH(2001), *Second-order optimality conditions for* $C^1$ multiobjective programming problems, Acta Mathematica Vietnamica, 26, 257-268.

M. GUIGNARD (1969), *Generalized Kuhn-Tucker conditions for mathematical programming problems in Banach spaces*, SIAM J. Control, 7, 232-241.

L. GUO, G.-H. LIN and J. J. YE (2013), *Second-order optimality conditions for mathematical programming with equilibrium constraints*, J. Optim. Theory Appl., 158, 33-64.

C. GUTIERREZ, B. JIMENEZ and V. NOVO (2009), *New second-order directional derivative and optimality conditions in scalar and vector optimization*, J. Optim. Theory Appl., 142, 85-106.

C. GUTIERREZ, B. JIMENEZ and V. NOVO (2010), *On second-order Fritz John-type optimality conditions in nonsmooth multiobjective programming*, Math. Programming, Ser. B, 123, 199-223.

O. GÜLER (2010), Foundations of Optimization, Springer, New York.

M. HACHIMI and B. AGHEZZAF (2007), *New results on second-order optimality conditions in vector optimization problems*, J. Optim. Theory Appl., 135, 117-133.

S. P. HAN and O. L. MANGASARIAN (1979), *Exact penalty functions in nonlinear programming*, Math. Programming, 17, 254-269.

H. HANCOCK (1960), Theory of Maxima and Minima, Dover, New York (originally published in 1917).

Y. HE and J. SUN (2011), *A weaker constraint qualification and a second-order necessary optimality condition for mathematical programming*, Pacific Journal of Optimization, 7, 19-28.

M. R. HESTENES (1966), Calculus of Variations and Optimal Control Theory, J. Wiley, New York.

M. R. HESTENES (1975), Optimization Theory: The Finite-Dimensional Case, J. Wiley, New York.

R. HETTICH and H. TH. JONGEN (1977), *On first and second order conditions for local optima for optimization problems in finite dimensions*, Methods of Operations Research, XXIII, Verlag Anton Hain, Meisenheim am Glan, 82-97.

A. HORSLEY and A. J. WROBEL (2003), *Liminal inequality constraints and second-order optimality conditions*, CDAM Research Report LSE-CDAM-2003-16.

A. HORSLEY and A. J. WROBEL (2004), *First and second-order conditions in constrained optimisation*, CDAM Research Report LSE-CDAM-2003-15.

N. Q. HUY and N. V. TUYEN (2016), *New second-order optimality conditions for a class of differentiable optimization problems*, J. Optim. Theory Appl., 171, 27-44.

A. D. IOFFE (1979), *Necessary and sufficient conditions for a local minimum, 3: second-order conditions and augmented duality*, SIAM J, Control and Opimization, 17, 266-288.

A. D. IOFFE (1989), *On some recent developments in the theory of second-order optimality conditions*; in S. Dolecki (Ed.), Optimization, volume 1405 of Lecture Notes in Mathematics, Springer, Berlin, 55-68.

A. D. IOFFE and V. M. TIKHOMIROV (1979), Theory of Extremal Problems, North Holland, Amsterdam.

V. I. IVANOV (2015), *Second-order optimality conditions for vector problems with continuously differentiable data and second-order constraint qualifications*, J. Optim. Theory Appl., 166, 777-790.

R. JANIN (1984), *Directional derivative of the marginal function in nonlinear programming. Sensitivity, stability and parametric analysis*, Mathematical Programming Study, 21, 110-126.

A. V. JITTORNTRUM (1984), *Solution point differentiability without strict complementarity in nonlinear programming*, Mathematical Programming Study, 21, 127-138.

B. JIMENEZ and V. NOVO (2002), *First and second order conditions for strict minimality in multiobjective programming*, Numer. Funct. Anal. Optim., 23, 303-322.

B. JIMENEZ and V. NOVO (2003), *Second order necessary conditions in set constrained differentiable vector optimization*, Math. Meth. Oper. Res., 58, 299-317.

B. JIMENEZ and V. NOVO (2004), *Optimality conditions in differentiable vector optimization via second order tangent sets*, Appl. Math. Optim., 49, 123-144.

F. JOHN (1948), *Extremum problems with inequalities as subsidiary conditions*; in K. O. Friedrichs, O. E. Neugebauer and J. J. Stoker (Eds.), Studies and Essays Presented to R. Courant on his 60th Birthday, Wiley, New York, 187-204. Reprinted in G. Giorgi and T. Kjeldsen (2014).

V. KALASHNIKOV, B. JADAMBA and A. A. KHAN (2006), *First and second order optimality conditions in set optimization*; in S. Dempe and V. Kalashnikov (Eds.), Optimization with multivalued mappings. Theory, applications and algorithms, Springer, New York, 265-276.

W. KARUSH (1939), Minima of Functions of Several Variables with Inequalities as Side Conditions, Master Thesis, Department of Mathematics, University of Chicago. Reprinted in G. Giorgi and T. Kjeldsen (2014).

H. KAWASAKI (1988a), *Second-order necessary conditions of the Kuhn-Tucker type under new constraint qualifications*, J. Optim. Theory Appl., 57, 253-264.

H. KAWASAKI (1988b), *An envelope-like effect of infinitely many ineqaulity constraints on second-order necessary conditions for minimization problems*, Math. Programming, 41, 73-96.

D. S. KIM and N. V. TUYEN (2018), *A note on second-order Karush-Kuhn-Tucker necessary optimality conditions for smooth vector optimization problems*, RAIRO Oper. Res., 52, 567-575.

M. KOJIMA (1980), *Strongly stable stationary solutions in nonlinear programs*; in S. M. Robinson (Ed.), Analysis and Computation of Fixed Points, Academic Press, New York, 93-138.

J. KYPARISIS (1985), *On uniqueness of Kuhn-Tucker multipliers in nonlinear programming*, Mathematical Programming, 32, 242-246.

J. KYPARISIS and A. V. FIACCO (1986), *Second-order sufficient conditions for strict and isolated local minima in nonlinear programming*, Technical Paper T-504, Department of Operations Research, George Washington University, Washington, D.C.

U. LEDZEWICZ and H. SCHAETTLER (1995), *Second-order conditions for extremum problems with non regular equality constraints*, J. Optim. Theory Appl., 86, 113-144.

E. S. LEVITIN, A. A. MILYUTIN and N. P, OSMOLOVSKII (1978), *Conditions of high order for a local minimum in problems with constraints*, Russian Math. Surveys, 33, 97-168.

B. LIU (1992), *Second-order sufficient conditions for isolated local minima*; Proceedings of the Second Conference of the Association of Asian-Pacific Operational Research Society (APORS) within IFORS, 509-516.

J. LIU and B.-G. LIU (1995), *On second-order sufficient conditions in smooth nonlinear programs*; in R. P. Agarwal (Ed.), Recent Trends in Optimization Theory and Applications, World Scientific, Singapore, 239-254.

S. LU (2011), *Implications of the constant rank constraint qualification*, Math. Programming, 126, 365-392.

S. LU (2012), *Relation between the constant rank and the relaxed constant rank constraint qualifications*, Optimization, 61, 555-566.

D. T. LUC (2002), *Second order optimality conditions for problems with continuously differentiable data*, Optimization, 51, 497-510.

D. LUENBERGER and Y. YE (2008), Linear and Nonlinear Programming, 3rd edition, Springer, New York.

M. C. MACIEL, S. A. SANTOS and G. N. SOTTOSANTO (2011), *On second-order optimality conditions for vector optimization*, J. Optim. Theory Appl., 149, 332-351.

M. C. MACIEL, S. A. SANTOS and G. N. SOTTOSANTO (2012), *On second-order optimality conditions for vector optimization: Addendum*, J. Optim. Theory Appl., DOI 10.1007/s10957-012-0029-2.

T. MAEDA (2004), *Second order conditions for efficiency in nonsmooth multiobjective optimization problems*, J. Optim. Theory Appl., 122, 521-538.

O. L. MANGASARIAN (1969), Nonlinear Programming, McGraw-Hill Book Co., New York.

H. MAURER (1981), *First and second order sufficient optimality conditions in mathematical programming and optimal control*, Math. Programming Study, 14, 163-177.

G. P. McCORMICK (1967), *Second-order conditions for constrained minima*, SIAM J. on Applied Mathematics, 15, 641-652. Reprinted in G. Giorgi and T. Kjeldsen (2014).

G. P. McCORMICK (1976), *Optimality criteria in nonlinear programming*; in R. W. Cottle and C. E. Lemke (Eds.), SIAM-AMS Proceedings, Vol. IX, American Mathematical Society, Providence, RI, 27-38.

G. P. McCORMICK (1983), Nonlinear Programming. Theory, Algorithms and Applications, J. Wiley, New York.

E. J. MESSERLI and E. POLAK (1969), *On second order necessary conditions of optimality*, SIAM J. Control, 7, 272-291.

L. MINCHENKO and A. LESCHOV (2016), *On strong and weak second order necessary optimality conditions for nonlinear programming*, Optimization, 65, 1693-1702.

L. MINCHENKO and S. STAKHOVSKI (2011), *On relaxed constant rank regularity conditions in mathematical programming*, Optimization, 60, 429-440.

E. NINH, W. SONG and Y. ZHANG (2012), *Second-order sufficient optimality conditions in vector optimization*, J. Glob. Optim., 54, 537-549.

C. D. PAGANI and S. SALSA (1990), Analisi Matematica. Volume 1, Masson, Milano. Analisis Matematica. Volume 2, Masson, Milano.

R. PALLU DE LA BARRIERE (1963), *Compléments à la théorie des multiplicateurs en programmation non linéaire*, Revue Française de Recherche Opérationnelle, 7, 163-180.

A. PASCOLETTI and P. SERAFINI (2007), *Differential conditions for constrained nonlinear programming via Pareto optimization*, J. Optim. Theory Appl., 134, 399-411.

L. PENNISI (1953), *An indirect proof for the problem of Lagrange with differential inequalities as added side conditions*, Trans. Amer. Math. Soc., 74, 177-198.

J.-P. PENOT (1994), *Optimality conditions in mathematical programming and composite optimization*, Math. Programming, 67, 225-245.

J.-P. PENOT (1998), *Second-order conditions for optimization problems with constraints*, SIAM J. Control Optim., 37, 303-318.

J.-P. PENOT (2000), *Recent advances on second-order optimality conditions*; in V. H. Nguyen, J.-J. Strodiot and P. Tossings (Eds.), Optimization, Lecture Notes in Economics and Mathematical Systems, Vol. 481, Springer-Verlag, Berlin, 357-380.

J.-P. PENOT (2008), *Softness, sleekness and regularity properties in nonsmooth analysis*, Nonlinear Anal., 60, 2750-2768.

J.-P. PENOT (2017), *Higher-order optimality conditions and higher-order tangent sets,* SIAM J. Optimization, 27, 2508-2527.

M. M. RIZVI and M. Nasser (2006), *New second-order optimality conditions in multiobjective optimization problems: differentiable case*, J. Indian Inst. Sci., 86, 279-286.

S. M. ROBINSON (1980), *Strongly regular generalized equations*, Mathematics of Operations Research, 5, 43-62.

S. M. ROBINSON (1982), *Generalized equations and their solutions, Part II: Applications to nonlinear programming*, Math. Programming Study, 19, 200-221.

R. T. ROCKAFELLAR (1989), *Second-order optimality conditions in nonlinear programming obtained by way of epi-derivatives*, Mathematics of Operations Research, 14, 462-484.

R. T. ROCKAFELLAR and R. J.-B. WETS (2009), Variational Analysis, Springer, New York.

S. ROLEWICZ (1983), *On sufficient conditions of optimality of second order*, Annales Polonici Mathematici, XLII, 297-300.

D. ROUX (1984), Lezioni di Analisi Matematica II, Masson, Milano.

A. RUSZCZYNSKI (2006), Nonlinear Optimization, Princeton University Press, Princeton.

A. SHAPIRO and F. AL-KHAYYAL (1993), *First-order conditions for isolated locally optimal solutions*, J. Optim. Theory Appl., 77, 189-196.

C. SHEN, W. XUE and Y. AN (2015), *A new result on second-order necessary conditions for nonlinear programming*, Oper. Res. Lett., 43, 117-122.

C. SIMON and L. BLUME (1994), Mathematics for Economists, Norton, New York. Italian version by A. Zaffaroni: Matematica per l'Economia e le Scienze Sociali (2 volls), EGEA, Milan.

G. STILL and M. STRENG (1996), *Optimality conditions in smooth nonlinear programming*, J. Optim. Theory Appl., 90, 483-515.

M. STUDNIARSKI (1986), *Necessary and sufficient conditions for isolated local minima of nonsmooth functions*, SIAM J. Control Optim., 24, 1044-1049.

A. TAKAYAMA (1985), Mathematical Economics, Cambridge Univ. Press, Cambridge.

T. TANINO, H. NAKAYAMA and Y. SAWARAGI (1978), *Multiplier functions and duality for nonlinear programmes having a set Constraint*, Int. J. Systems Sci., 9, 467-481.

A. A. TRETYAKOV (1984), *Necessary and sufficient conditions of optimality of p-order*, USSR Computational Mathematics and Mathematical Physics, 24, 123-127.

G. WACHSMUTH (2013), *On LICQ and the uniqueness of Lagrange multipliers*, Operations Reserach Lett., 41, 78-80.

S. Y. WANG (1991), *Second-order necessary and sufficient conditions in multiobjective programming*, Numer. Funct. Anal. Optim., 12, 237-252.

D. E. WARD (1993), *Calculus for parabolic second-order derivatives*, Set-Valued Analysis, 1, 213-246.

D. E. WARD (1994), *Characterizations of strict local minima and necessary conditions for weak sharp minima*, J. Optim. Theory Appl., 80, 551-571.